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FOUNDATIONS

Quine, W. V. Element and number. J. Symbolic Logic 6, 135–149 (1941).

The author is indebted to Rosser for a proof that the system of his "Mathematical Logic" [W. W. Norton, New York, 1940; these Rev. 2, 65], when the elementhood principle *200 is left out, is consistent since it would hold for a universe wherein the only entity is the null class. The author here recognizes the subsequent result of Rosser showing that the system when including *200 is inconsistent due to manifesting the Burali-Forti paradox. The essential revision called for seems to be achieved by reconstructing †610 of Chapter VI as an axiom (say †600), with a few obvious slight correlative changes elsewhere in that book. The author proceeds in this paper to an alternative development of the later sections. The suggestion is made of adopting what were called "counter sets" as the entities to be called natural numbers. Then (as with von Neumann) comparison with, rather than membership in, becomes the test for the number of a finite class. *A. A. Bennett.*

Rosser, Barkley. The Burali-Forti paradox. J. Symbolic Logic 7, 1–17 (1942).

It is shown that a contradiction is involved in the formal system described in W. V. Quine's book "Mathematical Logic" [W. W. Norton and Co., New York, 1940; these Rev. 2, 65] by deriving the Burali-Forti paradox (the one about the ordinal of all ordinals) in that system. The author's derivation may be described as a translation into the symbolism of Quine of the usual statement of the paradox, as given, for example, by Whitehead and Russell. This derivation consists of 44 theorems (the last of which is self-contradictory), numbered and derived as if they formed part of an eighth chapter of the book of Quine. The first 36 of these theorems are unobjectionable, and can be thought of as forming an introduction to a theory of ordinal numbers based on Quine's system when revised to avoid the paradox, as recently proposed by Quine [see the preceding review].

The last eight theorems are questionable, and fail to hold in Quine's revised system. The crucial theorem is †841, which states that the series of all ordinal numbers less than N has the ordinal number N. Rosser shows that, in many systems which avoid the paradox, this theorem fails to hold. The independent discovery of the paradox in Quine's system by R. C. Lyndon is acknowledged. *O. Frink.*

Rosser, Barkley. New sets of postulates for combinatory logics. J. Symbolic Logic 7, 18–27 (1942).

The author proposes a set of axioms for that form of the theory of combinators which he proved in his thesis [Ann. of Math. (2) 36, 127–150 (1935); Duke Math. J. 1, 328–355 (1935)] to be equivalent to the λ-formalism of Church [The Calculi of Lambda-Conversion, Princeton University Press, Princeton, N. J., 1941; see these Rev. 3, 129]. The new axiom set is more complex than the original, but it has the merit of making the derivation of the equivalence theorem vastly simpler. Since the equivalence theorem is the most important result in the theory of combinators, this is a notable advance. At the end of the paper the author sketches the application of his method to other forms of the theory of combinators, but the reviewer finds these considerations, especially those relating to systems with K present, obscure. Indeed, in the latter case the author has acknowledged the existence of a minor error. It is hoped this will be corrected later. *H. B. Curry* (Philadelphia, Pa.).

Curry, Haskell B. The combinatory foundations of mathematical logic. J. Symbolic Logic 7, 49–64 (1942).

This is the author's retiring presidential address of December, 1941 to the Association for Symbolic Logic. After giving a very clear exposition of the fundamentals of combinatory logic, showing its close relationship to the λ-calculi developed by Church and his students [The Calculi of Lambda-Conversion, Annals of Mathematics Studies, no. 6, Princeton University Press, Princeton, N. J., 1941; these Rev. 3, 129], the author describes the results of some of his more recent researches. These have to do on the one hand with simplified methods of deriving the paradoxes (such as those of Richard and Russell) in systems of logic which are inconsistent, and on the other hand with a method of introducing into combinatory logic undefined notions of generality, such as quantification or formal implication, in such a way that a consistency theorem like that of Church and Rosser is still derivable. The latter is an important advance, and is attained by the device of restricting by rule the application of the terms denoting generality to a certain category of so-called canonical terms, which include all terms intuitively denoting propositions or propositional functions. The paradox of Gödel is avoided, since the notion of canonicalness cannot be formalized in the system. It is indicated that these results may have an important bearing on the difficult question of the consistency of analysis.

Because of its nature this address contains no proofs, and only the briefest sketch of some of the more important points. The author is correct in describing his consistency theorem as remarkable, but he gives no indication of when or whether it will be published, although there are two footnotes which refer the reader to other as yet unpublished papers. *O. Frink* (State College, Pa.).

Bronstein, Daniel J. A correction to the sentential calculus of Tarski's Introduction to Logic. J. Symbolic Logic 7, 34 (1942).

It is shown that the postulates given on p. 147 of A. Tarski's book "Introduction to Logic and to the Methodology of Deductive Sciences" [Oxford University Press, New York, 1941; these Rev. 2, 209] are not, as was intended, adequate for the sentential calculus, but can be made so by any one of several slight changes. *O. Frink.*

Langford, C. H. Note on a device of Quine and Goodman. J. Symbolic Logic 6, 154–155 (1941).

The author explains a device used by Quine and Goodman in "Elimination of extra-logical postulates" [J. Symbolic Logic 5, 104–109 (1940); these Rev. 2, 65] and points out the objection that symbols having the same denotations such as "Scott" and "the author of Waverley" are not necessarily interchangeable for either of two familiar postulational purposes: the study of the consequences of an assumption with a view to testing its truth status, and the formulation of scientific findings in order to communicate these findings to others. *A. A. Bennett* (Aberdeen, Md.).

Newman, M. H. A. and Turing, A. M. A formal theorem in Church's theory of types. J. Symbolic Logic 7, 28–33 (1942).

The authors discuss the proof of Y^α (a form of the "axiom of infinity" for the type α) as used by Church. A proof was given by Church from Y^ι and Axioms 1 to 7 for all types of the forms ι', ι'', $\cdots$. This paper serves to clear up the question as to other types. It uses also Axioms 9 and 10. The three main steps may be roughly described as follows: (a) a proof that this "axiom of infinity" is equivalent to the proposition that no number belongs to its own posterity; (b) a proof that if a type τ can be mapped one-one on to a part of one of another type τ', then the axiom of infinity in τ' implies the same axiom in τ; (c) the actual construction of mappings for the pairs of types $\alpha\beta$, α and $\alpha\beta$, β. Constant use is made of Church's "deduction theorem." *A. A. Bennett* (Aberdeen, Md.).

Bernays, Paul. A system of axiomatic set theory. III. Infinity and enumerability. Analysis. J. Symbolic Logic 7, 65–89 (1942).

The author sketches the development of analysis on the basis of his axiomatic set theory, the foundations of which were given in parts I and II of this paper [J. Symbolic Logic 2, 65–77 (1937); 6, 1–17 (1941); these Rev. 2, 210]. His purpose is partly to show that analysis does not require the full strength of his axiom system. In particular, the axioms asserting the existence of the set which is the sum of a set of sets, and of the set of all subsets of a set, are not required, and only a weak form of the axiom of choice is needed. This is due to the fact that the only infinite sets which arise in analysis are enumerable. In cases where the power is nonenumerable, it suffices to deal with classes, such as the class of all real numbers. [The author's theory is based on the distinction between class and set due to J. von Neumann, only sets being eligible for membership. The purpose of this distinction is the avoidance of paradoxes.]

The theory of infinite classes and enumerable sets is first developed, based on the treatment of finite ordinals given in part II. Various forms of the axiom of infinity are discussed. Real numbers are introduced by means of the Dedekind cut. A method of representing continuous functions by enumerable sets is given. It is shown that the theory of Lebesgue measure and of Hilbert space may also be derived using only enumerable sets.

Much of this development of analysis does not even require the axiom of choice. Various special forms of this axiom, permitting only an enumerable number of acts of choice, are shown to suffice for certain familiar types of argument in analysis. One such special axiom of choice introduced by the author implies also the axiom of infinity. Finally it is shown that the same framework which is required for analysis is sufficient also for the theory of the ordinal numbers of the second number class. *O. Frink.*

McKinsey, J. C. C. A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. J. Symbolic Logic 6, 117–134 (1941).

The author solves the decision problem for the Lewis systems of strict implication S2 and S4 by showing that a sentence with r subsentences is provable in S2 (S4) if and only if it is satisfied by every finite normal S2-matrix (S4-matrix) with not more than $2^{2^{r+1}}$ (2^{2^r}) elements. This provides an effective decision procedure, since a method is given for constructing all finite normal S2- and S4-matrices. The proof, however, is noneffective, and depends on a nonconstructive method, due to Lindenbaum, of establishing the existence of characteristic matrices.

These results are applied to abstract topology by means of the correspondence, due to Tang Tsao-Chen, between the Lewis systems and topology, in which the Lewis $\Diamond$-operation corresponds to the topological operation of closure. It is shown that a sentence of S4 is provable if and only if the corresponding topological equation is true in every T-space. It follows from this and from known results in topology that there are just 14 distinct modalities in S4, but infinitely many distinct modal functions of one variable. The author shows that a topological equation of the form $A=1$ is true in every T-space if and only if it is true in every finite T-space with not more than 2^r points, where r is the number of subexpressions of A. He also shows that a topological equation is true in every T_1-space if and only if it is true in every T-space. One consequence of these results not mentioned by the author is that it is sufficient, in testing sentences of S4 by his decision procedure, to consider only those S4-matrices which have a single designated element. *O. Frink* (State College, Pa.).

Goodman, Nelson. Sequences. J. Symbolic Logic 6, 150–153 (1941).

The method of Wiener-Kuratowski for defining sequences on the basis of class theory alone yields as a sequence of length n a class n types higher than the components of the sequence. This author proposes a mode of definition applicable to sequences of any length, which gives in each case a class of only one type higher than the type common to its components. The ordered type (x, y, z) for example has as members all unit subclasses of x, all two-membered classes that are included in y, or that include y if y is not null, and all three-membered classes that are included in z or that include z if z is not null. Some classes are self-ordered, in that they establish sequences composed of their own members. Every class establishes some sequence, although not each class is a sequence. *A. A. Bennett.*

Gokieli, L. P. On so-called "content axioms" of mathematical logic. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 421–428, 665–672, 731–738 (1940); 2, 51–58 (1941). (Russian) [MF 5280]

By the term here translated "content axioms" the author refers to principles such as those of inference and substitution, which are now commonly referred to as rules of procedure. The paper contains an obscure and long winded philosophical discussion of such principles, which is of no direct interest for mathematicians. *H. B. Curry.*

Larguier, Everett H. Postulational methods. Scripta Math. 8, 99–109 (1941). [MF 6481]

An exposition for the uninformed as to the purpose of postulates in modern mathematics. *A. A. Bennett.*

Piaget, Jean. Le rôle de la tautologie dans la composition additive des classes et des ensembles. C. R. Soc. Phys. Genève 58, 102–107 (1941). [MF 6494]

Piaget, Jean. Le groupement additif des classes. C. R. Soc. Phys. Genève 58, 107–112 (1941). [MF 6495]

Piaget, Jean. Le groupement additif des relations asymétriques (sériation qualitative) et ses rapports avec le groupement additif des classes. C. R. Soc. Phys. Genève 58, 117–122 (1941). [MF 6496]

Piaget, Jean. Sur les rapports entre les groupements additifs des classes et des relations asymétriques et le groupe additif des nombres entiers. C. R. Soc. Phys. Genève 58, 122–126 (1941). [MF 6497]

Piaget, Jean. Les groupements de la classification complète et de l'addition des relations symétriques. C. R. Soc. Phys. Genève 58, 149–154 (1941). [MF 6498]

Piaget, Jean. Les groupements de la multiplication biunivoque des classes et de celle des relations. C. R. Soc. Phys. Genève 58, 154–159 (1941). [MF 6499]

Piaget, Jean. Les groupements de la multiplication counivoque des classes et des relations. C. R. Soc. Phys. Genève 58, 192–197 (1941). [MF 6503]

Piaget, Jean. La fonction régulatrice du groupement dans le développement mental: esquisse d'une théorie opératoire de l'intelligence. C. R. Soc. Phys. Genève 58, 198–203 (1941). [MF 6504]

The eight papers above, written by a psychologist, deal with a generalization of the notion of group, which the author believes to be of importance in psychology, and for which he employs the term groupement. Unfortunately, although many examples from the field of abstract sets and relations are given, the mathematical details of the theory are not made clear. *O. Frink* (State College, Pa.).

***Scholz, Heinrich. Metaphysik als strenge Wissenschaft.** Staufen-Verlag, Köln, 1941. 188 pp.

Dieses kleine, aber durchaus gehaltvolle Buch bietet dem Logiker und dem Philosophen ein doppeltes Interesse. Erstens wird eine formalisierte Theorie der Identität und Verschiedenheit entwickelt und zwar werden hintereinander die Syntax, die Semantik und die Axiomatik dieser Theorie aufgebaut. Die Menge der sinnvollen identitätstheoretischen Ausdrücke ist die kleinste Menge, der die identitätstheoretischen Atomausdrücke (das heisst, die Zeichenreihen der Gestalt "$x_i \equiv x_k$") angehören und die zugleich abgeschlossen ist in Bezug auf die aussagenerzeugenden Operationen (das heisst, Bildung der Negation, der Disjunktion, der Konjunktion, der Implikation und der Äquivalenz) und auf die Operationen der Quantifizierung (das heisst, Bildung der Generalisierte und der Partikularisierte). Identitätstheoretische Aussagen sind sinnvolle identitätstheoretische Ausdrücke, in denen keine freien Individuenvariablen vorkommen. Die planmässige Deutung dieses Formalismus ist eine Anwendung der semantischen Methode Tarskis. Es werden hintereinander die Begriffe der Erfüllung, der Wahrheit, der Allgemeingültigkeit, der Allgemeinungültigkeit und der Neutralität eingeführt. Dann können die Sätze der Identitätstheorie als allgemeingültige identitätstheoretische Aussagen definiert und die Begriffe der Gleichwertigkeit und Geltungsgleichheit umschrieben werden; letztere Begriffe stellen sich als umfangsgleich heraus. Nun werden die Zählaussagen, die Nicht-Zählaussagen und die ausgezeichneten Zählaussagen charakterisiert und es wird bewiesen: eine Zählaussage ist allgemeingültig dann und nur dann, wenn sie gleichwertig ist mit einer ausgezeichneten Zählaussage. Sodann werden die Begriffe der Anzahlaussage und der numerischen Aussage herangezogen und es wird bewiesen: eine identitätstheoretische Aussage (und a fortiori eine Zählaussage) ist neutral dann und nur dann, wenn sie gleichwertig ist mit einer numerischen Aussage. Bemerkenswert sind noch die hier nicht bewiesenen Sätze: es gibt ein allgemeines Verfahren, durch dessen Anwendung für jede identitätstheoretische Aussage effektiv entschieden werden kann, in welchen möglichen (das heisst, nicht leeren) Welten sie gültig ist; und: eine identitätstheoretische Aussage, die gültig ist im Bereich der natürlichen Zahlen, ist auch gültig in jeder Welt mit einer grösseren Individuenzahl.

Nunmehr wird die Axiomatisierbarkeit der Identitätstheorie bewiesen. An Axiomen wird ein Axiom der Reflexivität und ein Axiom der Drittengleichheit (für alle x_0, x_1, x_2 gilt: wenn $x_0 \equiv x_2$ und $x_1 \equiv x_2$, so $x_0 \equiv x_1$) zugrundegelegt, der Folgerungsbegriff wird bestimmt durch die Operationen der (vollfreien und gebundenen) Umbenennung, der Abtrennung, der (vorderen und hinteren) Generalisierung und der (vorderen und hinteren) Partikularisierung. Dann ist die Menge der identitätstheoretischen Sätze identisch mit der Folgerungsmenge einer Menge von identitätstheoretischen Aussagen, welche ausser den Axiomen noch diejenigen Aussagen umfasst, welche aus den Identitäten der Aussagenlogik durch Einsetzung hervorgehen.

Man findet also in diesem Buche ein vollständig und sorgfältig durchgearbeitetes Beispiel der Anwendung der neueren logischen Methoden an einer speziell zu diesem Zweck auserlesenen einfachen Theorie.

Zweitens wird nun gezeigt, in welchem Sinne diese Theorie als ein Probestück einer streng wissenschaftlichen Metaphysik aufgefasst werden kann. Die allgemeingültigen Aussagen bieten einen Überblick über die Gesamtheit der möglichen Welten. Diese Metaphysik ist eine Metaphysik im Sinne Leibnizens, der schon die Aufgabe gestellt hat "ewige und allgemeine Wahrheiten zu finden, so in allen Weltkugeln, ja . . . bei Gott selbst gelten müssen, von dem sie auch beständig herfliessen," und steht auf die Stufe einer philosophia perennis. *E. W. Beth* (Amersfoort, Netherlands).

***Thompson, D'Arcy Wentworth. On Growth and Form.** New edition. Cambridge University Press, Cambridge, England, 1942. x+1116 pp.

Table of contents: I. Introductory. II. On magnitude. III. The rate of growth. IV. On the internal form and structure of the cell. V. The forms of cells. VI. A note on adsorption. VII, VIII. The forms of tissues, or cell-aggregates. IX. On concretions, spicules, and spicular skeletons. X. A parenthetic note on geodetics. XI. The equiangular spiral. XII. The spiral shells of the foraminifera. XIII. The shapes of horns, and of teeth or tusks: with a note on torsion. XIV. On leaf-arrangement, or phyllotaxis. XV. On the shapes of eggs, and of certain other hollow structures. XVI. On form and mechanical efficiency. XVII. On the theory of transformations, or the comparison of related forms. Epilogue. Index.

The first edition appeared in 1916.

Loria, Gino. Dans quelles circonstances la découverte scientifique a-t-elle lieu? Univ. Nac. Tucumán. Revista A. 2, 357–368 (1941). [MF 6764]

Epstein, Paul S. The time concept in restricted relativity. Amer. J. Phys. 10, 1–6 (1942). [MF 6396]

ANALYSIS

Morse, Marston. What is analysis in the large? Amer. Math. Monthly **49**, 358–364 (1942). [MF 6790]

This is a paper which ranks high in this series of papers which have been appearing in the Monthly. In it the author mentions several problems and their solutions: continuous vector fields on a manifold, Liouville's theorem, Poincaré's work on differential equations and finally Morse's work on critical points. The reader is allowed to meet the problems and to understand the answers to them without going through the work leading to these answers, work involving the formidable arguments that have led most mathematicians to seek immunization against analysis in the large. [This review contains opinions of the reviewer and not necessarily those of the Naval Department or service.]

C. B. Tompkins (Washington, D. C.).

***Cartan, Henri. Sur les classes de fonctions définies par des inégalités portant sur leurs dérivées successives.** Actual. Sci. Ind., no. 867. Hermann et Cie., Paris, 1940. 36 pp.

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive (finite or infinite) numbers. If $f(x)$ is of class C^∞ in an interval I, it is said to belong to the class $\{A_n\}$ on I if for every x_0 in I there are positive numbers K and c such that $|f^{(n)}(x)| \leqq Kc^n A_n$ in a neighborhood of x_0. This monograph contains contributions to the problem of finding necessary and sufficient conditions for a given class $\{A_n\}$ to be contained in another given class $\{A_n'\}$. The problem was later completely solved by the author and Mandelbrojt, in a paper which has already appeared [Acta Math. **72**, 31–49 (1940); these Rev. **1**, 297]. Cartan's book also contains proofs of the inequalities for successive derivatives needed in the proofs of the conditions for the relations between classes [essentially equivalent results have also been given by Gorny, Acta Math. **71**, 317–358 (1939); these Rev. **1**, 137; see also Boas and Pólya, Duke Math. J. **9**, 406–424 (1942), in particular, p. 410; see the following review]. Cartan constructs examples showing that a class $\{A_n\}$ and its "rectified" class $\{\bar{A}_n\}$ may be distinct. He discusses the class of a compound function: if $\{B_n\}$ is a class such that $B_{n+1}/\{(n+1)B_n\}$ is a non-decreasing function of n, if $\{A_n\}$ is a class containing $\{B_n\}$, and if f and g are of classes $\{A_n\}$ and $\{B_n\}$, respectively, then $f(g)$ belongs to $\{A_n\}$ on any interval where it is defined. Extension of the fundamental results to functions of several variables is indicated. A number of other questions are discussed. In particular, the following condition is necessary and sufficient for a class $\{A_n\}$ to be identical with the class $\{n!\}$ of analytic functions: $\lim_{n\to\infty}\{A_n/n!\}^{1/n} \geqq 0$ and $\{A_{n_i}/n_i!\} \leqq M < \infty$ for a sequence $\{n_i\}$ such that n_{i+1}/n_i is bounded. *R. P. Boas, Jr.* (Chapel Hill, N. C.).

Boas, R. P., Jr. and Pólya, G. Influence of the signs of the derivatives of a function on its analytic character. Duke Math. J. **9**, 406–424 (1942). [MF 6878]

Detailed proof of a theorem, previously announced by these authors, which combines and generalizes the results of S. Bernstein and Widder in this field. [For a statement of their chief theorem 1 see Proc. Nat. Acad. Sci. U. S. A. **27**, 323–325 (1941); these Rev. **2**, 351.] The main point of the proof is the following new lemma 6: If p and q are positive integers and $g(x)$ is real-valued, possesses a $(p+2q)$th derivative, and satisfies $|g(x)| \leqq M$, $g^{(p+2q)}(x) \geqq 0$ in $[-1, 1]$, then $g^{(p)}(x) \leqq A^{p+2q}(p+2q)^p M$ in $[-1, 1]$, where $A = 30 \exp(30e+1)$ is an absolute constant. The paper is self-sufficient. All known results leading up to lemma 6, one of which is equivalent to a recent theorem of A. Gorny [Acta Math. **71**, 317–358 (1939); these Rev. **1**, 137], are provided with partly simplified proofs operating with Taylor approximating polynomials instead of polynomials of best approximation. *I. J. Schoenberg* (Philadelphia, Pa.).

Mandelbrojt, S. Analytic functions and classes of infinitely differentiable functions. Rice Inst. Pamphlet **29**, no. 1, 142 pp. (1942). [MF 6557]

The present lectures give an excellent account of the modern theory of classes of infinitely differentiable (i. d.) functions of a real variable and may be regarded as a second edition of the author's book "Séries de Fourier et classes quasi-analytiques de fonctions" [Gauthier-Villars, Paris, 1935; referred to below as M 1935] to include the work of the author and of H. Cartan since that date. For a brief review we may divide these lectures into five parts as follows. 1. (pages 1–75) Let M_n $(n=1, 2, \cdots)$ be a given sequence of positive constants and $I=[a, b]$ a given closed interval. Hadamard introduced the class $C\{M_n\}$ of all i. d. functions $f(x)$ defined in I such that to each of them there corresponds a constant $k=k(f)$ such that $|f^{(n)}(x)| < k^n M_n$ $(n \geqq 1, x \in I)$. The periodic class $C^*\{M_n\}$ is the subclass of those elements $f(x)$ of $C\{M_n\}$, for $I=[0, 2\pi]$, which, if extended mod 2π, are i. d. for $-\infty < x < \infty$. Finally the open class $C^0\{M_n\}$ is defined as the set of those i. d. $f(x)$ in an open interval $I^0=(a, b)$ and belonging to $C\{M_n\}$ in every closed subinterval $[\alpha, \beta]: a < \alpha < \beta < b$. Any one of these classes C is called analytic if its elements are analytic functions, and differentiable if $f \in C$ implies $f' \in C$. Solutions of the following major problems are here presented: (a) to find conditions on the sequence $\{M_n\}$ in order that $C\{M_n\}$ (or $C^*\{M_n\}$) be analytic; (b) conditions for the differentiability of $C^*\{M_n\}$; same for $C^0\{M_n\}$; (c) the problem of equivalence of classes proposed by Carleman in 1926 and solved by H. Cartan and the author [Acta Math. **72**, 31–49 (1940); these Rev. **1**, 297]: finding conditions that the classes $C^0\{M_n\}$ and $C^0\{M_n'\}$ be identical. The treatment is essentially that of M 1935 and the paper referred to. However, a systematic geometric and analytic exposition of the author's principle of regularization of sequences [M 1935, 95–96] provides a remarkably uniform foundation, the only other tools required being some elementary properties of Tchebycheff polynomials.

2. (pages 75–92) This part, devoted to the problem of quasi-analyticity of a class $C\{M_n\}$, gives a proof of the classical Carleman theorem in Ostrowski's formulation. The sufficiency proof is a modification of known proofs. The necessity proof is new and carried through by a construction of a function $f(x)$ in $C\{M_n\}$, which violates the quasi-analyticity of $C\{M_n\}$, by means of unpublished work of H. E. Bray on repeated mean-values of functions. 3. (pages 92–100) Exposition of the author's recent decomposition theorem [Acta Math. **72**, 15–29 (1940); these Rev. **1**, 297]. 4. (pages 101–118) This part, dealing with the author's work on quasi-analytic subclasses of $C\{M_n\}$ in $[0, 2\pi]$ whose elements have lacunary Fourier series, covers roughly chapters 7–9 of M 1935. More recent results by the author in collaboration with N. Wiener [C. R. Acad. Sci. Paris **203**, 34–36 (1936)] and with F. E. Ulrich [just appearing] are stated. 5. (pages 119–142) This last part, justifying the first part of the title of these lectures, deals with the problem of locating singularities of a power

series on the circumference of its circle of convergence. It contains two elegant theorems by the author [C. R. Acad. Sci. Paris **204**, 1456–1458 (1937); **206**, 730–732 (1938)] and an account of Denjoy's work [Ann. Ecole. Norm. (3) **55**, 257–336 (1938)] generalizing Pólya's applications of the indicator diagram to the location of singularities of power series. Let $f(z)=\exp\{-z^{-2}\}$, $z\neq 0$, $f(0)=0$. The author's proof on page 76 to the effect that $f(x)\epsilon C\{(2n)!\}$ in $[0,1]$ is invalid, since $f(z)$ is obviously unbounded in $|z-a|\leqq a$ $(0<a<1)$, but easily corrected. *I. J. Schoenberg.*

Perlin, Irwin E. Indefinitely differentiable functions in several real variables. Amer. J. Math. **64**, 441–446 (1942). [MF 6443]

The author has already obtained sufficient conditions [Bull. Amer. Math. Soc. **46**, 272–273 (1940); these Rev. **1**, 298] on a sequence $\{M_i\}$ in order that there exist a function $F(x)$ satisfying the condition l.u.b. $|F^{(i)}(x)|=M_i$, $0\leqq x\leqq a$. Corresponding results are obtained here for functions of several variables which are infinitely often differentiable.

A. C. Schaeffer (Stanford University, Calif.).

Widder, D. V. Completely convex functions and Lidstone series. Trans. Amer. Math. Soc. **51**, 387–398 (1942). [MF 6321]

A function $f(x)$ is said to be completely convex (c. c.) in the interval $a\leqq x\leqq b$ if it has there derivatives of all orders and if $(-1)^n f^{(2n)}(x)\geqq 0$ for $a\leqq x\leqq b$, $n=0, 1, 2, \cdots$. The author has previously shown that a function $f(x)$ satisfying these conditions can be extended analytically to an entire function [Proc. Nat. Acad. Sci. U. S. A. **26**, 657–659 (1940); these Rev. **2**, 219]. This result was established by means of Lidstone's series

$$(1)\qquad f(x)=\sum_{n=0}^{\infty}[f^{(2n)}(1)\Lambda_n(x)+f^{(2n)}(0)\Lambda_n(1-x)],$$

where the polynomials $\Lambda_n(x)$ are defined by the relations $\Lambda_0(x)=x$, $\Lambda_n''(x)=\Lambda_{n-1}(x)$, $\Lambda_n(0)=\Lambda_n(1)$, $n=1, 2, \cdots$. In the present paper the validity of (1) in the range $0\leqq x\leqq 1$ is investigated. It is shown that, if $f(x)$ is c. c. in $a\leqq x\leqq b$, where $b-a>1$, then $f(x)$ is entire of exponential type less than π and (1) holds uniformly in $[0,1]$. This sufficient condition for the validity of (1) in $[0,1]$ is not necessary. Both necessary and sufficient conditions may be stated in terms of the following definition: $f(x)$ is called a minimal completely convex function in $[0,1]$ if it is c. c. there and if $f(x)-\epsilon\sin\pi x$ is not c. c. in $[0,1]$ for any positive ϵ. It is then shown that (1) holds absolutely and uniformly in $[0,1]$ if and only if $f(x)$ is the difference of two minimal completely convex functions in $[0,1]$. *I. J. Schoenberg.*

Beckenbach, E. F. Convexity properties of generalized mean value functions. Ann. Math. Statistics **13**, 88–90 (1942). [MF 6387]

Let

$$\theta(t)=\left((x_2-x_1)^{-1}\int_{x_1}^{x_2}(f(t))^t dx\right)^{1/t},$$

in which $f(x)$ is positive and continuous in $[x_1, x_2]$, $-\infty<t<\infty$. Let $\lambda(t)=\log\theta(t)$. It is shown that these functions satisfy the differential inequalities $t\lambda''(t)\geqq -2\lambda'(t)$ and $t\theta''(t)\geqq t(\theta'(t))^2(\theta(t))^{-1}-2\theta'(t)$. The equality sign in one of these inequalities for even one value of t will imply $f(x)\equiv$ const. *I. J. Schoenberg* (Philadelphia, Pa.).

Konüs, A. On the theory of means. Acta [Trudi] Univ. Asiae Mediae. Ser. V-a., fasc. **24**, 10 pp. (1939). (Russian. English summary) [MF 5553]

Denote by $x_1, x_2, \cdots, x_n$ a set of observed values, x_i having the weight q_i, and let $Z=\sum_{i=1}^{n}qx_i$. Consider $I=I(x_1, x_2, \cdots, x_n)$ a certain "average" of the x_i. The author assumes that I is an analytic function and that $dZ=\sum_{i=1}^{n}q_i dx_i$. These assumptions, combined with some general analytical properties of average functions [Kolmogoroff, Nagumo], yield explicit expressions for the $q_i=q_i(x_1, x_2, \cdots, x_n)$. Applications are made to price index-numbers. *J. Shohat* (Philadelphia, Pa.).

Calculus

* **James, Glenn and James, Robert C. Mathematics Dictionary.** Digest Press, Van Nuys, Calif., 1942. v+259 pp.; appendix, 22 pp.

The content of this book is well described by its subtitle: "Giving the meaning of the basic mathematical words and phrases, including an exhaustive covering of all terms from arithmetic through the calculus and the technical terms commonly used in the applications of these subjects." The objective of the book is described in the introduction as follows: "The sustaining motive in the preparation of this dictionary has been somewhat pedagogical, namely the conveying of mathematical concepts. Formalism has throughout been a minor consideration. The needs of three classes of readers have been kept constantly in mind: (1) The student's need of a reliable supplement to his current work and his need of ready access, in the more advanced classes, to the multitude of facts that he has already studied. (2) The needs of teachers and workers in other fields that require mathematics. (3) The needs of the intelligent layman who wants to know how to answer the mathematical questions that confront him almost daily."

Users will certainly welcome a book of this kind. However, its usefulness might sometimes be impaired by serious errors, like: (1) "Abel's theorem on power series: if a power series in x converges for $x=c$, it converges absolutely for all values of x such that $-c<x<c$." (2) "Conjugate radicals: radicals of the form $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$; or, in general, $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$." *W. Feller.*

Gama, Lelio I. Note sur la démonstration du théorème de Rolle. Anais Acad. Brasil. Sci. **13**, 345–346 (1941). [MF 6590]

It is pointed out that B. Levi's recent proof of Rolle's theorem [Publ. Inst. Mat. Univ. Nac. Litoral **2**, 29–34 (1940); these Rev. **2**, 78] is the first "constructive" proof of the theorem; its advantages from a general point of view are stressed. *W. Feller* (Providence, R. I.).

Pompeiu, D. Sur le théorème des accroissements finis. Mathematica, Timişoara **17**, 129–136 (1941). [MF 6735]

This paper contains some elementary remarks on Rolle's theorem. Let Γ be a curve $y=f(x)$. Suppose that $f'(x)$, $f''(x)$ exist and let $y=ax^2+bx+c$ be the equation of a parabola Π passing through three points A, B, C of Γ. The author proves: There exists a translation T and a point M (on the arc ABC) such that T transforms Π into another parabola which possesses a contact of second or higher order with the given curve Γ at the point M. Among other remarks we

mention: (1) The converse of Rolle's theorem is not always true; it may happen that no chord is parallel to a given tangent. (2) Rolle's theorem is not always true for functions of a complex variable. *G. Fubini* (New York, N. Y.).

Azevedo do Amaral, Ignacio M. Note on the development into series of functions of one independent variable. Anais Acad. Brasil. Sci. **12**, 221–228 (1940). (Portuguese) [MF 6575]

For the expansion of a given function in a series of other functions a formula is presented which yields the series of Bürmann and Gomes Teixeira as special cases, and by more particular specialization such series as those of Taylor, Laurent and Fourier. The derivation of the general formula, by a method involving substitution of an appropriate expression for y in the Leibniz expansion of the nth derivative of the product $(x-1)^{-1}y$, provides a representation of the remainder term, but there is no discussion of convergence. *D. Jackson* (Minneapolis, Minn.).

Guarnieri, Angel J. On Riemann's function $\sum_1^\infty (nx)/n^2$ and $\sum_1^\infty (2^n x)/4^n$. Revista Union Mat. Argentina **7**, 135–139 (1941). (Spanish) [MF 6357]

The author computes the integral of Riemann's example of an integrable function of unbounded variation as well as that of a related function by integrating term by term and discusses numerically the series obtained [cf. Riemann, Gesammelte Mathematische Werke, 2nd ed., Teubner, Leipzig, 1892, p. 266]. *P. Scherk* (Bloomington, Ind.).

Balanzat, Manuel. On an application of the method of condensation of singularities. Revista Union Mat. Argentina **8**, 17–19 (1942). (Spanish) [MF 6967]

By the method indicated in the title a function is constructed which has a dense set of discontinuities and which is everywhere in an interval the derivative of another function. *J. V. Wehausen* (Columbia, Mo.).

Coe, C. J. Problems on maxima and minima. Amer. Math. Monthly **49**, 33–37 (1942). [MF 6143]

The author considers the problem of finding maxima and minima of a function $u(x, y)$ under an auxiliary condition $f(x, y)=0$. This is reduced to finding those points of the plane at which some curve of the family $u(x, y)=$constant is tangent to the curve $f(x, y)=0$. Several examples are given of maxima and minima problems which are easily solved by this method. *R. C. James* (Pasadena, Calif.).

Coronato, Savino. Criteri wronskiani di dipendenza lineare per funzioni di più variabili indipendenti. Pont. Acad. Sci. Comment. **5**, 283–318 (1941). [MF 5890]

This paper is concerned with establishing criteria for the linear dependence of m functions $u_i(x_1, \cdots, x_r)$ $(i=1, \cdots, m)$ of r variables $(x_1, \cdots, x_r)$ in a connected open region. The functions are assumed to have continuous partial derivatives of sufficiently high orders, and the criteria for linear dependence are phrased in terms of certain determinants (Wronskians) involving these functions and combinations of their partial derivatives. The author was evidently unaware of a paper by G. M. Green [Trans. Amer. Math. Soc. **17**, 483–516 (1916)], as the general criterion of the present paper is only a slight extension of Theorem II in this paper by Green. *W. T. Reid* (Chicago, Ill.).

van Veen, S. C. Riemann double integrals. Mathematica, Zutphen. B. **10**, 1–17 (1941). (Dutch) (to be continued) [MF 6940]

Coulson, C. A. Two-centre integrals occurring in the theory of molecular structure. Proc. Cambridge Philos. Soc. **38**, 210–223 (1942). [MF 6373]

This is a collection of 92 formulae for the calculation of the integral $J=\int r^n \rho^\nu e^{-br-\beta\rho} dV$, where r, ρ are bipolar coordinates, $n, \nu=-2, -1, 0, 1, 2$; b and β are constants greater than 0, and the integration is taken for the whole tridimensional space V. The formulae include also the case in which the integrand is multiplied by a product of first or second degree in $\sin\theta$, $\cos\theta$, θ being the angle between the radius vector r or ρ and the line joining the poles of the coordinate system. All formulae (with the exception of one) give J in terms of the exponential function and the exponential integral. The possibility of calculating J by means of the incomplete gamma function is also indicated. *I. Opatowski* (Chicago, Ill.).

Fréchet, Maurice. Les courbes d'inertie et l'ajustement. Portugaliae Math. **3**, 69–78 (1942). [MF 6372]

The author studies the analogues of ellipses of inertia and axes of inertia of a plane material system if in the expression $\sum m_i d_i^2$ of the moment of inertia of the system with respect to a line Δ we replace the squares d_i^2 of the distances by the distances $|d_i|$ themselves. The discussion of the case of continuous mass distributions is preceded by an elementary geometric and analytic investigation of the case of a finite number of material points. *I. J. Schoenberg.*

Theory of Series

Bradshaw, J. W. The modification of an infinite product. Quart. J. Math., Oxford Ser. **12**, 216–220 (1941). [MF 6547]

Infinite products are modified in order to increase the rapidity of convergence by multiplying the nth partial product by an appropriate rational function of n, in a manner analogous to the author's method of modifying infinite series [Amer. Math. Monthly **46**, 486–492 (1939); these Rev. **1**, 50]. Application is made to Wallis' product for $\pi/2$. *P. W. Ketchum* (Urbana, Ill.).

Oguiewetzki, I. On Dirichlet's test for uniform convergence. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 441–444 (1941). (Russian. English summary) [MF 6834]

The author proves the theorem: A necessary and sufficient condition for the uniform convergence of the series $\sum_{i=0}^\infty \alpha_i(x)u_i(x)$ within a certain region M, where $\sum u_i(x)$ oscillates between finite limits for all points x of M, is that $\alpha_i(x)$ tends to zero uniformly at all points x of M and that $\sum_{i=0}^\infty |\alpha_i(x)-\alpha_{i+1}(x)|$ is uniformly convergent in M. *František Wolf* (Berkeley, Calif.).

Hornich, Hans. Über beliebige Teilsummen absolut konvergenter Reihen. Monatsh. Math. Phys. **49**, 316–320 (1941). [MF 6854]

Let $a_1+a_2+\cdots$ be a series of positive terms for which $\sum a_i=1$; the problem is that of determining properties of the set N of numbers x representable in the form $x=\sum' a_j$, where $\sum'$ is a finite sum or infinite series of terms a_j with different subscripts. Suppose the terms ordered so that $a_1 \geqq a_2 \geqq a_3 \geqq \cdots$ and let $\sigma_k=\sum_{j=k+1}^\infty a_j$, $k=1, 2, 3, \cdots$. The set N is a perfect subset of the interval $0 \leqq x \leqq 1$. If, for a

fixed index k, $\sigma_k<a_k$, then the interval $\sigma_k<x<a_k$ belongs to the complement of N; extensions of this result are given. If, for each k, $\sigma_k<a_k$, then the set N contains no interval and the measure of N is equal to the limit of the decreasing sequence $2^k\sigma_k$. The set N is the entire interval $0\leqq x\leqq 1$ if and only if $a_k\leqq\sigma_k$ for each k. If $a_n=2^{-n}$, then N is the entire interval $0\leqq x\leqq 1$; and if $a_n=2/3^n$, then N is the Cantor middle-third set. *R. P. Agnew* (Ithaca, N. Y.).

Hadwiger, H. Über die Konvergenzarten unendlicher Reihen im Hilbertschen Raum. Math. Z. **47**, 325–329 (1941). [MF 6719]

Let $\sum_{r=1}^{\infty}a_r$ be a convergent series of elements of Hilbert space. The author defines six types of convergence as follows: The series $\sum_r a_r$ is called absolutely convergent (briefly, AC) if the series $\sum_r|a_r|$ converges; nonabsolutely convergent (NAC) if it is not absolutely convergent; conditionally convergent (CC) if the existence of the sum depends on the order of the elements; unconditionally convergent (UC) if it is not conditionally convergent; conditional sum (CS) if the sum depends on the order of the elements; unconditional sum (US) if it is not a conditional sum. If X and Y denote any two types of convergence, the symbol $X\rightarrow Y$ means that any X-series is a Y-series, but not conversely; and $X\leftrightarrow Y$ means that any X-series is a Y-series and also conversely. The author proves the following relations: AC$\rightarrow$UC; AC$\rightarrow$US; NAC$\leftarrow$CC; NAC$\leftarrow$CS; UC$\rightarrow$US; CC$\leftarrow$CS. In a finite dimensional space the above six relations remain valid if we replace $\rightarrow$ by $\leftrightarrow$. *A. Wald.*

Banach, S. Sur la divergence des séries orthogonales. Studia Math. **9**, 139–155 (1940). (French. Ukrainian summary) [MF 5263]

Let F be the space formed of all normal orthogonal sequences $\varphi_i(t)$ in the interval $(0, 1)$. The distance of two sequences $\varphi_i(t)$ and $\psi_i(t)$ is defined by

$$[\varphi_i(t), \psi_i(t)]=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{\|\varphi_i(t)-\psi_i(t)\|}{1+\|\varphi_i(t)-\psi_i(t)\|},$$

where $\|\varphi(t)\|=(\int_0^1\varphi^2(t))^{\frac{1}{2}}$. The space F is clearly metric, complete and separable. The author proves the following theorems: I. The set of complete sequences $\varphi_i(t)$ form a G_δ which is everywhere of the second category. II. Let $\sum_{i=1}^{\infty}c_i^2<\infty$; then either $\sum_{i=1}^{\infty}c_i\varphi_i(t)$ converges almost everywhere for all sequences of F, or the set of sequences of F for which almost everywhere

$$\overline{\lim}\,|\sum_{i=1}^{n}c_i\varphi_i(t)|=+\infty$$

is a G_δ which is everywhere of the second category. III. Let E be a subset of (L^2). Suppose E is the sum of a denumerable number of compact sets. Consider the set R of sequences $\varphi_i(t)$ of F such that for each $f(t)$ of E we have $(f(t)\neq 0)$

$$\overline{\lim}\,|\sum_{i=1}^{n}\varphi_i(t)\int_0^1 f(t)\varphi_i(t)dt|=\infty$$

almost everywhere. Then R is a G_δ which is everywhere of the second category. *P. Erdös* (Philadelphia, Pa.).

Soddy, F. The summation of infinite harmonic series. Proc. Roy. Soc. London. Ser. A. **179**, 377–386 (1942). [MF 6399]

Harmonic series with alternating signs are transformed into more rapidly converging series. The device to accomplish this result is based on the formal property of such series that, if any $n+1$ successive terms are multiplied by the binomial coefficients of the nth power of a binomial, the result can be summed into one term of simple form, namely, a constant divided by the product of the $n+1$ denominators of the terms. The terms of the given series are grouped into successive groups of $n+1$ terms each, then multiplied by the binomial coefficients, and each group summed by this principle. Finally an averaging process is performed over this sum and the similar sums obtained by shifting all the groups a certain number of spaces to the right. The result is particularly simple in case the series runs from $-\infty$ to $+\infty$. Various special cases are considered in which very rapidly converging series are obtained for π, $\log_e 2$ and other constants. *P. W. Ketchum* (Urbana, Ill.).

Nichols, G. D. A sufficient condition for Cesàro summability. Bull. Amer. Math. Soc. **48**, 580–582 (1942). [MF 7053]

The author proves a theorem closely related to a theorem of M. S. MacPhail [Bull. Amer. Math. Soc. **47**, 483–487 (1941); these Rev. **3**, 148]. *T. Fort* (Bethlehem, Pa.).

Hill, J. D. Some properties of summability. Duke Math. J. **9**, 373–381 (1942). [MF 6875]

Let A, A' and A'' denote, respectively, the three transformations

$$t_m=\sum_{k=1}^{\infty}a_{m,k}s_k,\quad t_m'=\sum_{k=1}^{\infty}a_{m,k-1}s_k,\quad t_m''=\sum_{k=1}^{\infty}a_{m,k+1}s_k,$$

in which $a_{m,0}=0$ for each m. The transformation A permits omission and adjunction of elements if and only if $\lim_{m\to\infty}a_{mk}=0$ for each k, and the two transformations A' and A'' include A. For the case in which A has an inverse A^{-1}, the conditions of Mazur are applied to characterize (in terms of the matrices of A and A^{-1}) the transformations permitting omission and adjunction of elements. Application is made to transformations of the form

$$a_{m,k}=p_k/(p_1+\cdots+p_m).$$

R. P. Agnew (Ithaca, N. Y.).

Szász, Otto. Some new summability methods with applications. Ann. of Math. (2) **43**, 69–83 (1942). [MF 6062]

The author investigates a method of summability defining the sum of a series $\sum a_k$ as $\lim_{n\to\infty}\sum_{k=0}^{n}a_kz_n^k$, where $\{z_n\}$ is a given sequence of complex numbers. Necessary and sufficient conditions are given which the sequence $\{z_n\}$ must satisfy if the method is to be regular. The theorems are extended to similar methods of summation, for example to the one defining the sum of $\sum a_k$ as $\lim_{n\to\infty}\sum_{k=0}^{n}a_k\rho_n^k\cos k\theta_n$, where $\{\rho_n\}$ and $\{\theta_n\}$ are sequences of real numbers. The results are applied to the theory of Fourier series.

A. Zygmund (South Hadley, Mass.).

McFadden, Leonard. Absolute Nörlund summability. Duke Math. J. **9**, 168–207 (1942). [MF 6348]

Let p_n be a sequence with nonvanishing partial sums P_n and let $t_n=\sum_{k=0}^{n}(p_{n-k}/P_n)U_k$. A sequence U_k is summable by the Nörlund method N_p if the transform t_n is convergent, and is absolutely summable N_p (summable $|N_p|$) if the transform t_n has bounded variation. Obviously $|N_p|\subset N_p$. An example is given to show that, if $p_n=1$ for each n so that N_p is the Cesàro transformation C_1, then $|N_p|\not\supset N_p$. [That $|N_p|\not\supset N_p$ for each Nörlund method is readily shown by use of the fact that N_p has an inverse.] By use of a theorem of Miss Mears [Ann. of Math. (2) **38**, 594–601 (1937)],

necessary and sufficient conditions for $|N_q| \subset |N_p|$ are obtained. Simpler sufficient conditions are given. A theorem of Kogbetliantz (that $|C_\alpha| \subset |C_\beta|$ when $-1<\alpha\leqq\beta$) is proved, and sufficient conditions for $|C_k| \subset |N_p|$, k an integer not less than 0, are given. If $p_n=p^n$ and $q_n=q^n$, where $0<q<p\leqq 1$, then $|N_q| \subset |N_p|$. If $q_n=q^n$, $0<q<1$, and $p_n=1/(n+1)$, then $|N_q| \subset |N_p|$; and $|N_p| \subset |C_\alpha|$ for each $\alpha>0$. If q_n is nonnegative and nondecreasing and q_{n+1}/q_n is monotone increasing (or decreasing) then $|N_q| \subset |C_1|$ (or $|C_1| \subset |N_q|$). Relations between $|N_p|$ and absolute Abel summability $|A|$ are given.

If $0<\alpha\leqq 1$ and $f(x)$ is a function, of period 2π, belonging to class Lip α, then the Fourier series and the conjugate Fourier series of $f(x)$ are both summable $|N_p|$ provided $p_n\geqq 0$, p_n is nonincreasing, $p_n\to 0$, $p_{n+1}-p_n$ is nonincreasing, $\sum(P_k/k)^2<\infty$, and $\sum k^{-\alpha-1}P_k^{-1}<\infty$. A similar theorem applies to functions in class Lip (α, q), that is, to functions $f(x)$ for which

$$\left\{\int_a^b |f(x+h)-f(x)|^q dx\right\}^{1/q}\leqq A|h|^\alpha.$$

Several corollaries of these theorems are given; they include some of the more recent as well as classical theorems on absolute summability of Fourier series by convergence and other methods of summability. *R. P. Agnew.*

Rogosinski, W. W. On Hausdorff's methods of summability. Proc. Cambridge Philos. Soc. **38**, 166–192 (1942). [MF 6252]

Let $\{t_n^{(1)}\}$ and $\{t_n^{(2)}\}$ be the transforms associated with two Hausdorff methods (H-methods) of summation T_1 and T_2, respectively. If the convergence of $\{t_n^{(1)}\}$ implies the convergence of $\{t_n^{(2)}\}$, the author writes $T_2\supseteq T_1$. In the case that there is at least one sequence $\{s_n\}$ such that $\{t_n^{(1)}\}$ diverges while $\{t_n^{(2)}\}$ converges, we write $T_2\supset T_1$. If $T_2\supseteq T_1$ and $T_1\supseteq T_2$, we write $T_1\sim T_2$. A method T is called regular (convergence preserving) if $T\supseteq C$, where C denotes convergence. Associated with each H-method of summation T is a moment sequence $\{\mu_n\}$ and a Mellin transform or a moment function $T(z)$,

$$\mu_n=\int_0^1 t^n d\phi(t), \quad T(z)=\int_0^1 t^z d\phi(t).$$

T is regular if and only if $\phi(t)$ is of bounded variation on the closed interval $(0, 1)$; additional conditions for consistency with C are that $\phi(t)$ be continuous at $t=0$ and $\phi(1)-\phi(0)=1$. Hausdorff proved that if $T_2=MT_1$, where M is regular, then $T_2\supseteq T_1$, and the converse result provided that none of the $\mu_n^{(1)}$ vanishes. The author establishes the converse result whenever at most a finite number of the $\mu_n^{(1)}$ vanish. The author also derives nine theorems, some with applications, pertaining to the strength of a regular H-method which depend largely on the zeros of its moment function in the closed half-plane $\Re(z)\geqq 0$, the point $z=\infty$ included. In this connection the relative strength of two regular H-methods T_1 and T_2 is seen to depend primarily on whether the quotient $M(z)$ of their respective moment functions $T_1(z)$ and $T_2(z)$ is a Mellin transform or not. Extensive use is made of the results of H. R. Pitt, in particular, whose work was concerned mostly with problems of the Tauberian and Mercerian type. *H. L. Garabedian* (Evanston, Ill.).

Garten, V. Über den Vergleich der Cesàroschen und Hölderschen Mittelbildungen. Math. Z. **47**, 111–124 (1940). [MF 6302]

Let $c^{(r)}(x)$ and $h^{(r)}(x)$ denote, respectively, the Cesàro and Hölder integral transforms of a function $s(x)$. For the case in which r is a positive integer, relations are given between the superior limits, the inferior limits and the oscillations of the two transforms $c^{(r)}(x)$ and $h^{(r)}(x)$. Equivalence of the integral transformations is a corollary of the results obtained. The results are analogous to results obtained by C. E. Winn and others for the Cesàro and Hölder sequence-to-sequence transformations. *R. P. Agnew.*

Knopp, Konrad. Über eine Klasse konvergenzerhaltender Integraltransformationen und den Äquivalenzsatz der C- und H-Verfahren. Math. Z. **47**, 229–264 (1941). [MF 6821]

In § 2 [5 pp.] is given an exposition of the algebra of multiplicative matrix transformations of sequences, with special reference to Hölder and Cesàro transformations H_r and C_r and to the Hurwitz-Silverman-Hausdorff family of transformations which commute with the arithmetic mean transformation M and with each other. In § 3 [12 pp.] there is a similar development of an algebra for kernel transformations of the form

$$\sigma(x)=\int_0^\infty V(x, t)s(t)dt \tag{1}$$

as applied to functions $s(t)$ bounded and integrable (Lebesgue) over each finite interval $0\leqq t\leqq a$. The kernel $V(x, t)$ is assumed to satisfy a set of conditions sufficient [but not necessary; see R. P. Agnew, Bull. Amer. Math. Soc. **45**, 689–730 (1939); these Rev. 1, 50] to make (1) multiplicative. Let E denote the identity transformation (not a kernel transformation), and let A and B denote kernel transformations of the form (1). While A cannot have an inverse of the form B, a transformation of the form $E-A$ may have an inverse of the form $E-B$. The transformations $E-A$ and $E-B$ are inverses if and only if A and B commute and $A+B=AB$. Any two transformations with kernels of the form $x^{-1}f(t/x)$ commute. Using these facts, the author shows that, if p is a constant not 0 and if $V(x, t)=pxt^{-2}$ and $W(x, t)=px^{p+1}t^{-p-2}$ when $0<x<t$, while $V(x, t)=W(x, t)=0$ otherwise, then $E+V$ and $E-W$ are inverses. In § 4 [8 pp.] the identity

$$(p+1)MC_p-pMC_{p+1}=C_{p+1}, \qquad \Re p>0, \tag{2}$$

is proved both for kernel and for matrix transformations. Equivalence of C_k and H_k for each $k=1, 2, 3, \cdots$ is then proved simultaneously for kernel and matrix transformations. The proof is accomplished by obtaining explicit formulas for regular transformations P_k and Q_k such that $H_k=P_kC_k$ and $C_k=Q_kH_k$. Comparisons with earlier proofs are given. In § 5 [3 pp.] it is shown that some of the formulas of § 4 may be obtained by straightforward application of standard methods of solving Volterra integral equations. In § 6 [3 pp.] it is shown that some of the transformations previously considered belong to a class of transformations neither weaker nor stronger than convergence-preserving transformations. In § 7 [2 pp.] are given formulas for transformations R_k and S_k ($\Re k>0$) such that $A=R_kC_k$ and $A=R_kH_k$, where A is an Abel transformation. The Abel transform $\sigma(x)$ of a function $s(t)$ is the integral over $0<t<\infty$ of $x^{-1}e^{-t/x}s(t)$. *R. P. Agnew.*

Good, I. J. Some relations between certain methods of summation of infinite series. Proc. Cambridge Philos. Soc. 38, 144–165 (1942). [MF 6251]

A function $f(t)$ which is not equal to zero on a set of positive measure in each interval $0 \leqq t_0 \leqq t < \infty$ and which has nonvanishing moments

$$c_n = \int_0^\infty f(t) t^n dt, \qquad n = 0, 1, 2, \cdots,$$

the integrals being Cauchy-Lebesgue integrals, defines a method of summability f which evaluates a series $\sum a_n$ to S if $\varphi(s) \to S$ as $s \to \infty$, where

$$\varphi(s) = \int_0^\infty f(t) \sum_{n=0}^{\infty} a_n c_n^{-1} t^n dt.$$

A method f is effective for $\sum a_n$ if it evaluates $\sum a_n$; it is applicable to $\sum a_n$ if the transform $\varphi(s)$ of $\sum a_n$ exists, that is, if $\sum a_n c_n^{-1} t^n$ is an entire function. Let $f(t) = c(\exp -t^c) t^{\gamma}$, $g(t) = d(\exp -t^d) t^{\gamma}$, where $d \geqq c > 0$ and $\delta \geqq \gamma > -1$. The inequalities insure that f is at least as widely applicable as g. It is shown that, if g is applicable to $\sum a_n$ and f is effective, then g is effective. The author points out that we have here a situation in which the Hardy-Littlewood "empirical principle of summability" is valid: If f is more widely applicable than g, and if g is applicable and f effective, then g is effective. *R. P. Agnew* (Ithaca, N. Y.).

Scott, W. T. and Wall, H. S. The transformation of series and sequences. Trans. Amer. Math. Soc. 51, 255–279 (1942). [MF 6317]

This paper is concerned primarily with Hausdorff and Gronwall methods of summation. Associated with each Hausdorff method $(H, \phi(u))$ or (H, c_p) is a moment sequence $\{c_p\}$, where $c_p = \int_0^1 u^p d\phi(u)$, $\phi(u) \epsilon BV(0, 1)$. Associated with each Gronwall method (f, g) is a mapping function $z = f(w)$ and a weight function $g(w) = \sum_{p=0}^{\infty} b_p w^p$, $b_p \neq 0$. In part I of the paper the authors consider the problem of determining conditions upon a sequence of functions $\{\beta_p(u)\}$ in order that the sequence $\{c_p\}$, where $c_p = \int_0^1 \beta_p(u) d\phi(u)$, be a moment sequence for every $\phi(u)$ in $BV(0, 1)$, and of determining conditions under which (H, c_p) is essentially regular. With this theory infinite classes of methods of summation are obtained all including a given mean or all equivalent to a given mean. In part II all methods of summation which are common to the class of Hausdorff methods and the class of Gronwall methods are determined. These are found in the method (H, c_n), where $c_n = \beta^n / C_{n+\alpha, n}$, $0 < \beta \leqq 1$, $\alpha \geqq 0$, the Gronwall method identical with this Hausdorff method being (f, g), where

$$f(w) = \beta w / (1 - (1-\beta) w), \quad g(w) = (1-w)^{-\alpha-1}.$$

In part III properties of a method of summation arising from an algorithm of Schur are studied. In particular, this method, shown to be a Gronwall method, affords a characterization of a class of real power series bounded in the unit circle. *H. L. Garabedian* (Evanston, Ill.).

***Garabedian, H. L. and Wall, H. S. Topics in continued fractions and summability.** Northwestern University Studies in Mathematics and the Physical Sciences, no. 1: Mathematical Monographs, vol. 1, pp. 87–132. Graduate School, Northwestern University, Evanston, Ill., 1941. $2.25.

This represents a summary of results obtained in recent years by the writers and others concerning certain aspects of the theory of algebraic continued fractions (a bibliography of 16 titles is appended). The topics covered are as follows: Hausdorff's methods of summation and analytic continuation, some continued fraction transformations yielding various identities, also the theory of Stieltjes' continued fractions in relation to the problem of moments and to totally monotonic sequences. *J. A. Shohat.*

Paydon, J. Findlay and Wall, H. S. The continued fraction as a sequence of linear transformations. Duke Math. J. 9, 360–372 (1942). [MF 6874]

The continued fraction

$$(1) \qquad 1 + \frac{a_2|}{|1} + \frac{a_3|}{|1} + \frac{a_4|}{|1} + \cdots,$$

in which $a_2, a_3, a_4, \cdots$ are complex numbers, is regarded as the infinite product of the linear transformations $t_1(v) = v$, $t_n(v) = 1/(1 + a_n v)$, $n = 2, 3, 4, \cdots$. If V_c is the circular region $|z - c| \leqq |c|$, $\Re(c) = 1$, it is shown that $t_n(V_c) \subset V_c$ if and only if a_n lies in the region A_c bounded by the parabola:

$$(2) \qquad \rho = \frac{\frac{1}{2}\cos^2 \frac{1}{2}\alpha}{1 - \cos(\theta - \alpha)}, \quad -\pi < \alpha < \pi; \ \Im(c) = -\tan \tfrac{1}{2}\alpha.$$

If $a_2, a_3, a_4, \cdots$ are in A_c it then follows that all the approximants of (1) are on the interior of V_c. It is then shown that if, in particular, $a_2, a_3, a_4, \cdots$ lie in any bounded closed region entirely within A_c then (1) converges. Thus, if a is an arbitrary point not on the real interval $-\infty < x \leqq -1/4$, then there exists a neighborhood of a of arbitrarily large (finite) area which is a convergence region for (1). From their main result the authors derive the convergence theorem of Stieltjes [Perron, Die Lehre von den Kettenbrüchen, 2nd ed., Teubner, Leipzig, 1929, p. 403] for the continued fraction

$$(3) \qquad 1 + \frac{a_2 z|}{|1} + \frac{a_3 z|}{|1} + \frac{a_4 z|}{|1} + \cdots,$$

where a_n is real and positive, and a theorem of Van Vleck [Perron, p. 288] for (3) in the case that $\lim a_n$ is a finite (complex) number. If in (1), $|a_n| \leqq t \leqq 1/4$, $n = 2, 3, 4, \cdots$, there is determined a circular region W_t in which the value of (1) must lie, and which is filled by these values. If w is a point on the boundary of W_t, there is one and only one continued fraction with $|a_n| \leqq t$ whose value is w. In the final section there is an elementary treatment of (1) in the case where $a_{n+1} = (1 - g_{n-1}) g_n x_n$, $0 \leqq g_n \leqq 1$; $n = 1, 2, 3, \cdots$; $g_0 = 0$; somewhat sharpened forms of a theorem of Pringsheim and a theorem of Van Vleck [Perron, § 53] are proved and the two theorems are shown to be equivalent to one another. *H. L. Garabedian* (Evanston, Ill.).

Functional Equations

Ghermanescu, Michel. Sur quelques équations fonctionnelles linéaires. Bull. Soc. Math. France 68, 109–128 (1940). [MF 6810]

The author derives the complete solution of the functional equation

$$\alpha \cdot \varphi(x, y, z) + \beta \cdot \varphi(y, z, x) + \gamma \cdot \varphi(z, x, y) = f(x, y, z)$$

for a function φ. Cyclic permutation of x, y, z produces a system of linear equations for φ. Solutions exist and are uniquely determined, unless $\alpha = \beta = \gamma$ or $\alpha + \beta + \gamma = 0$; in the

exceptional cases f has to satisfy certain functional equations. The author obtains his results by what amounts to a study of the group of permutations of three elements x, y, z.

F. John (New York, N. Y.).

Ghermanescu, M. Sur une classe d'équations fonctionnelles. C. R. Acad. Sci. Paris **211**, 199–201 (1940). [MF 5344]

The note announces that the functional equation

$$(1) \qquad \sum_{i=1}^{n} A_i \varphi(x_i, x_{i+1}, \cdots, x_n, x_1, \cdots, x_{i-1}) = 0,$$

with at least two coefficients different from zero, admits a nontrivial solution if the equations

$$(2) \qquad 1 - x^n = 0, \quad \sum_{i=1}^{n} A_i x^{i-1} = 0$$

have a common root. If $1-x^n = D(x)\sum_{i=1}^{p} B_i x^{i-1}$, with $D(x)$ the greatest common divisor of the left members of the equations (2), the general solution of (1) is given by

$$\varphi = \sum_{i=1}^{p} B_i u(x_i, x_{i+1}, \cdots, x_n, x_1, \cdots, x_{i-1}),$$

with the function u arbitrary. Some particular cases are cited, and an application to the equation obtainable from (1) by replacing the right member by $f(x_1, x_2, \cdots, x_n)$ is given.

R. E. Langer (Madison, Wis.).

Galbura, Giorgio. Sopra una certa equazione funzionale. Pont. Acad. Sci. Acta **5**, 7–14 (1941). [MF 4882]

The author studies the question of existence of solutions φ of the functional equation

$$(1) \qquad \varphi(x+y) - \varphi(x) - \varphi(y) = f(x, y).$$

He finds, as necessary and sufficient condition for the existence of twice differentiable solutions of (1), that $f(x, y)$ is a twice differentiable, symmetric solution of a differential equation of the form

$$(2) \qquad \frac{\partial^2 f(x, y)}{\partial x \partial y} = \psi(x+y),$$

or, that $f(x, y)$ is of the form $\int_0^x du \int_0^y \psi(u+v)dv + a$, with constant a and integrable ψ. If the assumption of differentiability of φ is dropped, the necessary condition (2) is replaced by an analogous difference equation for f, which, for example, is satisfied by all functions $f(x, y)$ symmetric to the origin.

F. John (New York, N. Y.).

Jackson, F. H. q^δ equations and Fibonacci numbers. Quart. J. Math., Oxford Ser. **12**, 211–215 (1941). [MF 6546]

The q-functional equations

$$[q^{2\delta} + (q^\delta/qx) - (1/qx^2)]y = 0, \quad [q^{2\delta} + (q^\delta/qx) - 1/qx^2]\eta = 0,$$

in which q^δ denotes the operator $\exp[(\log q)xd/dx]$, satisfy the conditions necessary for the existence of an equation of the third order with solutions of type $Y = y\eta$ [see Quart. J. Math., Oxford Ser. **11**, 1–17 (1940); these Rev. **2**, 133]. This equation is in fact

$$[x^6 q^{3\delta+6} - 2x^4 q^{2\delta+2} - 2x^2 q^\delta + 1]Y = 0.$$

The substitution

$$y = \sum_{n=-\infty}^{\infty} A x^n, \quad Y = \sum_{n=-\infty}^{\infty} B_n x^{2n}$$

leads to the result that A_n and B_n are expressible in terms of the Fibonacci numbers F_n which are defined by means of the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $F_2 = 1$. It is found that in one case the series can be expressed as the sum of four Jacobian functions $AI(w_1^2 x^2) + AI(w_2^2 x^2) + BI(w_1 x^2) + BI(w_2 x^2)$, where w_1 and w_2 are the roots of the equation $w^2 = w + 1$ and $I(z)$ is Jacobi's function

$$I(z) = \sum_{n=-\infty}^{\infty} q^{n^2-n} z^{2n}.$$

H. Bateman (Pasadena, Calif.).

Isaacs, Rufus Philip. A finite difference function theory. Univ. Nac. Tucumán. Revista A. **2**, 177–201 (1941). [MF 6758]

The author develops the properties of two types of functions, each type satisfying a pair of difference equations analogous to the Cauchy-Riemann differential equations.

P. Franklin (Cambridge, Mass.).

Gercevanoff, N. Sur quelques applications des itérations fractionnaires et complexes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 835–836 (1941). [MF 5510]

In this note the author states the definition of real and complex iterations of a function. He uses the iteration idea to find a solution of some particular examples of the functional equation (a) $f(\varphi(x)) = \Phi(x)$ (φ, Φ given), and makes some assertions regarding the application of iteration to the general equation (a), especially for the case where $\varphi(\varphi(x)) \equiv x$. These assertions are in part unclear, in part dubious.

I. M. Sheffer (State College, Pa.).

GEOMETRY

Rosenblatt, Alfred. On some inequalities in elementary geometry. Actas Acad. Ci. Lima **4**, 156–161 (1941). (Spanish) [MF 6612]

Rosenblatt, Alfred. On some elementary inequalities for the tetrahedron. Actas Acad. Ci. Lima **4**, 213–230 (1941). (Spanish) [MF 6613]

In the first paper the author offers a very natural and elementary synthetic proof of the first two propositions considered by G. N. Watson [Quart. J. Math., Oxford Ser. **11**, 273–276 (1940); these Rev. **2**, 294], and also a synthetic, but much longer, discussion of Watson's third theorem. The paper concludes with an analogous inequality for the tetrahedron, presumed to be true.

In the second paper this presumption is submitted to a searching analysis, mostly by very elementary synthetic means. Several results are obtained, of which the following may be quoted here. In a tetrahedron $ABCD$ in which $BC=a$, $CA=b$, $AB=c$, $DA=d$, $DB=e$, $DC=f$ and $d, e \geqq c$; $e, f \geqq a$; $f, d \geqq b$, we have $h \geqq a\sqrt{\frac{2}{3}}$, where h is the altitude to the face ABC and a is the side of the triangle ABC that is not greater than either of the other two sides.

N. A. Court (Norman, Okla.).

Court, N. A. On the anharmonic associates of a point for a triangle and a tetrahedron. Bol. Mat. **15**, 25–28 (1942). [MF 6773]

Buchanan, H. E. On poristic quadrilaterals. Amer. Math. Monthly **49**, 364–371 (1942). [MF 6791]

Jacobi's method, using elliptic functions, for obtaining the necessary and sufficient condition for a polygon to be

inscribed in a circle (R) and circumscribed about a second circle (r) is applied to the case of a quadrilateral (A). The author also uses elliptic functions to derive some properties of the diagonals of (A) and to prove further that the quadrilateral determined by the points of contact of the sides of (A) with the circle (r) is also poristic. *N. A. Court.*

Rossier, Paul. Sur les ovales de Descartes. C. R. Soc. Phys. Genève **58**, 95–96 (1941). [MF 6493]

From the well-known optical properties of a Cartesian oval (foci F', F'', equation $r'+nr''=a$) the author derives several geometrical properties of the curve. Thus he gives the following ruler and compass construction of the tangent to the curve at a given point P. With P as center draw two circles having for radii r'' and nr''. Draw the tangent to the second circle at the point where this circle is met by $F'P$ produced beyond P. This tangent cuts the tangent to the first circle at the point F'' in a point belonging to the required tangent. *N. A. Court* (Norman, Okla.).

Rossier, Paul. Sur la théorie des poloconiques et sa généralisation. C. R. Soc. Phys. Genève **58**, 207–209 (1941). [MF 6505]

Generalizing the notion of the poloconic of a line with respect to a cubic, the author shows analytically that the locus of a point whose polar conic with respect to a curve of degree n touches a given line is a curve of degree $2(n-2)$. It is pointed out as a consequence that a curve of degree n contains $2n(n-2)$ parabolic points, that is, points whose polar conics with respect to the given curve are parabolas. *N. A. Court* (Norman, Okla.).

Godeaux, Lucien. Sur les surfaces du quatrième ordre, circonscrites à deux tétraèdres de Moebius. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 472–486 (1941). [MF 6897]

Godeaux, Lucien. Sur une configuration géométrique dans l'espace à quatre dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 183–188 (1941). [MF 6904]

Steck, Max. Die Geometrie der erweiterten Konfigurationen. I. Eine Erweiterung des Konfigurationsbegriffs. Univ. Nac. Tucumán. Revista A. **2**, 331–356 (1941). [MF 6763]

Tables showing the incidences of elements represented by rows with those represented by columns are discussed and exemplified. The utility of such tables in establishing an axiomatic basis for geometry is pointed out. *J. L. Dorroh* (Baton Rouge, La.).

de Oliveira Júnior, Ernesto Luiz. Note on Desargues' triangles. Anais Acad. Brasil. Sci. **13**, 97–98 (1941). (Portuguese) [MF 6583]

The note is devoted to showing that a necessary and sufficient condition that two triangles in the relation of Desargues shall determine a perspectivity or a homology between their planes is that the point of intersection of the lines joining corresponding vertices of the triangles be not a vertex of the triangles. *L. M. Blumenthal.*

Merz, K. Einseitige Polyeder nach Boy. I. Einseitiges Hexadekaeder. Comment. Math. Helv. **14**, 134–137 (1942).

Humbert, Pierre. Einseitige Polyeder nach Boy. II. Polyèdre sans singularités topologiques homéomorphe au plan projectif dans l'espace à 3 dimensions. Comment. Math. Helv. **14**, 137–140 (1942).

As a topological model of the real projective plane the one-sided heptahedron fails in two respects. It crosses itself (along its three "axes"), and its six vertices are singular (so that it is not a manifold but only a pseudo-manifold). The same remark applies to the closely related quartic surface of Steiner. In 1901 W. Boy described a more satisfactory model: a surface which crosses itself along certain curves but otherwise is everywhere regular. [See Hilbert and Cohn-Vossen, Anschauliche Geometrie, Springer, Berlin, 1932, pp. 266–283.] The present authors obtain polyhedral varieties of Boy's surface by modifying the heptahedron. Both the new polyhedra have 16 faces. Merz's has 18 vertices and 33 edges. Humbert's has 24 vertices and 39 edges. *H. S. M. Coxeter* (Toronto, Ont.).

Fano, Gino. Sui cerchi ortogonali a due cerchi dati. Univ. Nac. Tucumán. Revista A. **2**, 87–94 (1941). [MF 6751]

An article of Bertrand Gambier [Bull. Sci. Math. **63**, 233–238 (1939); these Rev. **1**, 157] and the paradox involved is discussed by the author and extended in several ways. *N. A. Court* (Norman, Okla.).

Hummel, Frank H. A method of inversion. Math. Gaz. **26**, 73–80 (1942). [MF 6666]

An inversion is considered in which two corresponding points are collinear with the center of inversion, and the invariant points of the transformation lie not on a circle, but on two (real horizontal) parallel lines equidistant from the center of inversion. The fundamental theorems of the transformation are: (1) The inverse of a straight line is a parabola with a horizontal axis, passing through the center of inversion. (2) The inverse of a parabola with a horizontal axis is a parabola with a horizontal axis. This "parabolic" inversion is used by the author to solve problems involving parabolas in much the same way as the more familiar "circular" inversion is used in questions involving circles. *N. A. Court* (Norman, Okla.).

Mandan, Ram. Umbilical projection. Proc. Indian Acad. Sci., Sect. A. **15**, 16–17 (1942). [MF 6638]

Using umbilical projections [H. F. Baker, Principles of Geometry, Cambridge University Press, Cambridge, England, 1925, vol. 4, chaps. 1 and 2], the author derives properties of the centers of similitude and of the circles of antisimilitude of two circles in the plane. These properties are also obtained for three and four circles taken two at a time. The propositions obtained for four circles are applied to the special case when these circles pass through four given points taken three at a time. *N. A. Court.*

Mehmke, Rudolf. Geometrie der Lage oder des Masses? Monatsh. Math. Phys. **49**, 303–311 (1941). [MF 6852]

The author points out that there is a certain danger of losing a general outlook if the theorems of metric geometry are always considered as subordinated to those of projective geometry. In order to illustrate his statement, he gives a few examples (theorem of Desargues, projectivity of pencils of points, complete quadrilaterals) where the analytic proofs of projective theorems contain metric relations, which can often be expressed in terms of mechanics, and which may sometimes continue to have sense even in special cases when the projective theorem becomes meaningless. *E. Helly.*

Terracini, Alejandro. Origins of some geometrical concepts. Publ. Inst. Mat. Univ. Nac. Litoral **3**, 157–199 (1941). (Spanish. French summary) [MF 6740]

The author distinguishes between those developments that are empirical, based on physical objects, and those

which are abstractions, without exterior bases. The former are called discoveries, the latter inventions. In the first part, the history of differential geometry is traced from the introduction of the concepts of tangency, curvature (total, mean), geodesics and congruences of lines, emphasizing the use of geometric figures and proceeding, at first empirically, to rigorous mathematical concepts, through Euler, Dupin, Monge and Lagrange to Gauss. Among the important concepts of the second kind are featured the cross-ratio, as developed by Pappus, the fundamental group, as presented by Klein in the Erlangen Programm. Both of these ideas are applied to the projective definition of element of arc of an analytic curve, resulting in the Mehmke-Segre invariant. In conclusion, the author proposes a generalization of this idea, as a natural extension of the metric definition [Terracini, Period. Mat. (4) **13**, 99–119 (1933)]. *V. Snyder.*

Kasner, Edward. A notation for infinite manifolds. Amer. Math. Monthly **49**, 243–244 (1942). [MF 6512]

This puts on record the notation

$$_{\infty}r_0f(0)+r_1f(1)+\cdots+r_kf(k)$$

for a manifold depending on r_0 constants, r_1 functions of one variable, $\cdots$, and r_k functions of k variables.

P. Franklin (Cambridge, Mass.).

Bock, H. Fehlerwahrscheinlichkeit bei geometrischen Konstruktionen. Z. Phys. Chem. Unterricht **54**, 41–44 (1941). [MF 6858]

Boys, Charles V. Square ruled paper projection. J. Sci. Instruments **19**, 65–71 (1942). [MF 6700]

Practical directions are given for drawing the orthogonal projection of a space figure from various points of view, by plotting calculated distances on ruled paper rather than by the use of plans and elevations. *P. W. Ketchum.*

Krames, Josef. Zur Ermittlung eines Objektes aus zwei Perspektiven. (Ein Beitrag zur Theorie der "gefährlichen Örter.") Monatsh. Math. Phys. **49**, 327–354 (1941). [MF 6856]

It is a well-known theorem [proved by G. Hauck, J. Reine Angew. Math. **95**, 1–35 (1883)] that two perspectives of any object together with their interior orientation are sufficient to reconstruct the object. This problem of reconstruction has generally a unique solution. The author discusses particular cases, of some interest in photographic surveying, where the problem has more than one solution. He shows that there are two (or three) essentially different solutions if the points given in both perspectives and the two centers of projection lie on an orthogonal ruled surface of second order and if at the same time one (both) pairs of generators passing through these centers are adjoint generators. Any quadric surface generated by two congruent pencils of planes is called an orthogonal ruled surface of second order. Any two generators which can be used as axes of congruent generating pencils are called adjoint generators.

E. Lukacs (Baltimore, Md.).

Aschenbrenner, C. Die Lichtverteilung in Luftbildern. Bildmessung und Luftbildwesen. Beilage Allg. Vermessgs.-Nachr. **16**, 5–19 (1941). [MF 6784]

Ott, Harry G. The contribution to mapping errors of various imperfections in the construction and manipulation of multiplex equipment. Photogrammetric Engrg. **8**, 110–120 (1942). [MF 7092]

Lense, J. Bemerkungen zur Lehre von den Kartenentwürfen. Luftfahrtforschung **17**, 251–255 (1940). [MF 6561]

Considerations of elementary differential geometry are used to give a precise theory of the mapping of parts of a sphere on the plane. The fundamental directions ("Hauptrichtungen") and the indicatrix of Tissot are introduced and their importance is illustrated by the discussion of area preserving mappings. A special paragraph is devoted to the projection of Bonne. *O. Neugebauer* (Providence, R. I.).

Grambow, J. B. Die Lambertsche Karte als Luftnavigationskarte. Luftfahrtforschung **17**, 281–290 (1940). [MF 6562]

It has been proposed to replace the Mercator projection by the Lambert conic projection for use in aerial navigation ["The Lambert projection, its advantages for airway maps in the U. S. A.," Field Engineers Bulletin, ed. by the Coast and Geodetic Survey, Dec., 1935]. The author shows, however, that the Mercator projection remains superior for all practical purposes of long distance navigation. The deviations between the exact curves and the approximations useful in practice are investigated both theoretically and numerically. *O. Neugebauer* (Providence, R. I.).

Lambert, Walter D. The distance between two widely separated points on the surface of the earth. J. Washington Acad. Sci. **32**, 125–130 (1942). [MF 6913]

Convex Domains, Integral Geometry

Dinghas, Alexander. Neuer Beweis eines Satzes von Wirtinger und Blaschke. Math. Z. **47**, 265–274 (1941). [MF 6822]

The following theorem is proved: If $u(x_1, \cdots, x_n)$ is of class C' on the unit-sphere S in $x_1 \cdots x_n$-space, and if $\int_S u dS = 0$, then

$$J_n(u) = \int_S [(n-1)u^2 - \nabla u] dS \leqq 0,$$

where the equality sign only holds if u coincides with a linear function on S. Here ∇u denotes Beltrami's first differential operator. This theorem occupies a central position in the proof of Minkowski's inequality $W_{n-1}^2 \geqq W_n W_{n-2}$ for mixed volumes [Dinghas, Nachr. Ges. Wiss. Göttingen. Fachgruppe I (N.F.) **1**, 213–235 (1940); these Rev. **1**, 265; Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, nos. 4, 9, 11, 17; these Rev. **2**, 261–262]. The lemma (for $n=2, 3$ due to Wirtinger and Blaschke, respectively) had been proved previously by the author, using expansions in terms of generalized spherical harmonics for spheres in n-space. In the present paper the author succeeds in proving the lemma in a direct and considerably more elementary way. He uses induction over the dimension n; a recurrence formula is derived, expressing J_n in terms of integrals J_{n-1} over spheres in $(n-1)$-dimensional space; the lemma is then proved directly for $n=2$. The author applies the lemma to the derivation of a number of known inequalities for mixed volumes. *F. John* (New York, N. Y.).

Robinson, Chas. V. Spherical theorems of Helly type and congruence indices of spherical caps. Amer. J. Math. **64**, 260–272 (1942). [MF 6430]

Values of m such that the intersection of each m of a class of convex subsets of the n-dimensional sphere of

radius r implies a point common to all members of the class are found as follows: $m=n+k+2$ if one member of the class contains no k-dimensional sphere of radius r, $0\leqq k\leqq n$; $m=2n+1$ if the class contains more than $2n+2$ members; $m=n+1$ if each member of the class is of arc diameter less than $2\pi r/3$. As an extension of the covering properties of spherical caps (of the ordinary sphere of radius r), "complete sets of congruence indices" in the sense of L. M. Blumenthal [Bull. Amer. Math. Soc. 47, 435–443 (1941); these Rev. 3, 138] with respect to the class of all semimetric spaces are given for (1) the sphere; (2) the hemisphere; (3) a cap of radius ρ, $r\cdot\arccos(1/3)\leqq\rho<\pi r/2$; (4) a cap of radius ρ, $\rho<r\cdot\arccos(1/3)$. They are: (1) [5, 0]; (2) [5, 1]; (3) [4, 2], [5, 0]; (4) [4, 1]. Attention is called to case (4) in which the complete set requires two indices.

J. L. Dorroh (Baton Rouge, La.).

Gericke, H. Zur Relativ-Geometrie ebener Kurven. Math. Z. 47, 215–228 (1941). [MF 6820]

This paper deals with the isoperimetric inequality in "relative" geometry, developed by W. Süss [Jap. J. Math. 4, 57–75 (1927)]. In that geometry all quantities are defined relative to a basic convex curve E ("Eichkurve"). If $k(\varphi)$ is the function of support of E [that is, $k(\varphi)$ = distance from origin of tangent forming angle $\varphi+\pi/2$ with the x-axis], and if $h(\varphi)$ is the function of support of any convex curve C, the R-function of support of C is defined by $\gamma=h/k$. Usually γ is referred to the R-angle ψ as variable, ψ being defined by $d\psi/d\varphi=k\cdot r$, where r is the radius of curvature of E. The total R-length of the closed curve C is given by $L=\int_0^{2p}\gamma(\psi)d\psi$, where $2p$ is the R-length of E. Areas F are defined as in ordinary geometry.

The proof of the isoperimetric inequality $L^2/4p-F\geqq 0$ is constructed on the model of Hurwitz' well-known proof of the ordinary isoperimetric inequality. Instead of using Fourier series, $\gamma(\psi)$ is here expanded into a series of eigenfunctions of the equation $d(k^3rdy/d\psi)/d\psi+\lambda\psi=0$, with $y(2p)=y(0)$, $y'(2p)=y'(0)$. The essential point is that all positive eigenvalues are not less than 1. An upper bound for $L^2/4p-F$ is obtained in analogy to an inequality given by Dinghas [Nachr. Ges. Wiss. Göttingen. Fachgruppe I (N.F.) 1, 213–235 (1940); these Rev. 1, 265]. *F. John.*

Fiala, F. Le problème des isopérimètres sur les surfaces ouvertes à courbure positive. Comment. Math. Helv. 13, 293–346 (1941).

The principal purpose of the author is to generalize the isoperimetric inequality $L^2\geqq 2A\cdot 2\pi$ between the length L of a simply closed curve C in the Euclidean plane and the area A enclosed by it. F. Bernstein and E. Schmidt had already solved the problem for curves on the sphere and in the hyperbolic plane, respectively. In addition to these cases in which the surfaces containing the curves C are of constant Gauss curvature, the problem was solved for surfaces of negative Gauss curvature by Beckenbach and Radó [Trans. Amer. Math. Soc. 35, 662–674 (1933)]. The author investigates the problem for certain open surfaces S of nonnegative Gauss curvature. The surfaces S are complete abstract analytic Riemann surfaces of dimension two which are homeomorphic with the plane. The important notion of completeness [cf. Hopf and Rinow, Comment. Math. Helv. 3, 209–225 (1931)] is defined by the requirement: every bounded sequence of points (bounded in the sense of the metric) must have at least one limit point. The author establishes the inequality

$$L^2\geqq 2A(2\pi-K_C), \tag{1}$$

where L is the length of a curve C, A the area of the surface enclosed by it, and K_C the integral of the Gauss curvature over the same area. Since the Gauss curvature K is by hypothesis nonnegative, an inequality

$$L^2\geqq 2A(2\pi-K_T), \tag{2}$$

which is independent of the curve C, can then be obtained, where K_T is the integral of K over the entire surface. The right hand side of (2) is never negative, since all surfaces S have been shown by Cohn-Vossen [Compositio Math. 2, 69–133 (1935)] to have a total Gauss curvature $K_T\leqq 2\pi$. It is shown that the equality sign holds only for $K\equiv 0$ and C a "circle." For a fixed surface it is shown that a sequence of closed curves C_n exists such that

$$\lim_{n\to\infty} L^2(C_n)/A(C_n)=2(2\pi-K_T).$$

Hence the coefficient of A in (2) is the largest possible. The author considers also the question whether for given L there exists an upper bound for A for all curves C of length L. This is shown to be the case if $K_T<2\pi$, but for $K_T=2\pi$ such a bound exists only if $L<L^*$, where L^* depends on S. Finally it is shown that any surface S has infinite total area, a result by no means obvious for abstract surfaces of the kind considered here.

The proof of (1) is carried out by a method analogous to that of Crone and Frobenius [reproduced in Blaschke, Vorlesungen über Differentialgeometrie, Springer, Berlin, 1930, p. 56] for the case of the plane. To do so, the "parallels" to a simply closed curve C are considered. Such parallels can be defined in the neighborhood of C as the orthogonal trajectories of the geodesics normal to C. On a surface of positive curvature, however, such a system of curves can not be extended to form a set of coordinate curves on the whole surface, which is what is desired. The difficulty is overcome by introducing the notion of "true parallel" to C at distance p from C. By this is meant the locus of points at distance p from C, the distance from a point P of S to C being defined as the g.l.b. of the lengths of all curves joining P with the points of C. On the assumption that the surface is analytic and that C is a regular analytic curve, it is shown that a true parallel to C is composed of a finite number of regular analytic arcs each of which belongs to a geodetic parallel to C, and is made up of a finite number of closed curves. In addition, it is shown that the true parallels may be used as coordinate curves since through any point of S there passes one and only one such parallel. The proof of (1) is based on the possibility of embedding C in such a set of curves. More than half of the paper is devoted to establishing these and other properties of the true parallels. Some of these results are generalizations of results of Myers [Duke Math. J. 1, 376–391 (1935)] which refer to what might be called "true circles," that is, loci of points at constant distance from a fixed point rather than a fixed curve. In the author's proofs extensive and essential use of the hypothesis of analyticity is made.

J. J. Stoker (New York, N. Y.).

Balanzat, Manuel. Sur quelques formules de la géométrie intégrale des ensembles dans un espace à n dimensions. Portugaliae Math. 3, 87–94 (1942). [MF 6772]

This paper contains n-dimensional generalizations of previous results of the author [C. R. Acad. Sci. Paris 210, 596–598 (1940); Union Mat. Argentina, Publ. no. 14 (1940); these Rev. 2, 155]. *P. Scherk* (Bloomington, Ind.).

Raimondi, Elba R. A problem of geometric probabilities on the sets of triangles. Revista Union Mat. Argentina **8**, 1–16 (1942). (Spanish) [MF 6966]

The set of all the triangles (mod the group of similarities) permits one-to-one mappings on the points of various domains in the Euclidean plane or of surfaces in 3-space. The author discusses some of these mappings and computes in each case the Euclidean areas of the whole domain and of the domain upon which the set of all the acute (obtuse) triangles is mapped. Their quotient (which, of course, depends upon the mapping) is called the probability for a triangle to be acute (obtuse). *P. Scherk.*

Hajós, Georg. Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter. Math. Z. **47**, 427–467 (1941). [MF 6728]

In this paper Minkowski's conjecture on the boundary case of his theorem on homogeneous linear forms is proved. We define a cube with center $(a_1, \cdots, a_n)$ in Euclidean n-space R_n as the set of points $(x_1, \cdots, x_n)$ with $-\frac{1}{2} \leqq x_\nu - a_\nu < \frac{1}{2}$ $(\nu = 1, \cdots, n)$, and we define a k-covering of R_n as a set of cubes such that each point of R_n belongs to precisely k of them and such that their centers form a lattice L. Then Minkowski's conjecture can be formulated geometrically as the special case $k=1$ of the following hypothesis: The boundaries of two suitable cubes of a k-covering of R_n have an $(n-1)$-dimensional cube in common.

The author first proves by a method similar to that of Perron's proof of a theorem of Keller [Math. Z. **46**, 1–26 (1940), in particular, pp. 6–12; these Rev. **2**, 153] that we can assume the coordinates of all the lattice points to be rational ($k \geqq 1$). Then L can be considered as a sublattice of another one L' which is generated by certain points $(1/m_1, 0, \cdots, 0), (0, 1/m_2, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1/m_n)$, where $m_1, m_2, \cdots, m_n$ are rational integers [of course, L' is not uniquely determined]. The group of L' (mod L) is a finite Abelian group G. If L belongs to a 1-covering, there are n elements $A_1, \cdots, A_n$ of G and n positive integers $a_1, \cdots, a_n$ such that each element A of G can be written in one and only one way in the form $A = \alpha_1 A_1 + \cdots + \alpha_n A_n$ $(0 \leqq \alpha_\nu < a_\nu;\ \nu = 1, \cdots, n)$. Minkowski's conjecture is equivalent to the (purely group theoretical) contention that $a_\kappa A_\kappa = 1$ for a suitable κ. Replacing n by a larger number, this theorem can readily be reduced to the case that all the a's are prime numbers. The remainder of its proof is complicated and (19 pages) long. The more general conjecture is proved to be true for $n \leqq 3$ and all k, but false for $n \geqq 4$ and suitable $k > 1$. *P. Scherk.*

Algebraic Geometry

Perron, Oskar. Über das Vahlensche Beispiel zu einem Satz von Kronecker. Math. Z. **47**, 318–324 (1941). [MF 6825]

It is known that the points of any algebraic curve in 3-space can be obtained as the intersection of four or less algebraic surfaces. Vahlen [J. Reine Angew. Math. **108**, 346–347 (1891)] stated, without giving complete proof, that a rational space quintic curve with a quadrisecant could not be obtained as the intersection of fewer than four surfaces. In the present paper it is shown that this curve can be obtained as the intersection of three quintic cones. The question is therefore still open as to whether there exists a curve which is not the complete intersection of three, or even two, surfaces. *R. J. Walker.*

Fubini, Guido. On Abel's converse theorem. Ann. of Math. (2) **43**, 471–500 (1942). [MF 7003]

Starting from a classical property of plane cubics of genus 1, namely, that the elliptic parameters u, v, w of any three collinear points satisfy $u+v+w=0$, the author raises the converse problem as follows. Let three plane arcs be given, respectively, by their parametric equations $M=M(u)$, $N=N(v)$, $P=P(w)$, and let them be such that the points $M(u)$, $N(v)$, $P(w)$ are collinear (in a neighborhood of $u=v=w=0$) if and only if $u+v+w=0$; what can then be said about the three arcs? The problem is reduced, first, to an equation in the coordinates of $M(u)$, $N(v)$ and their first and second derivatives, and then to an equation of the first order. The conclusion is that the three arcs must be arcs of one and the same cubic, which may be of genus 1 or degenerate to genus 0 or to a reducible curve (either a conic and a straight line, or three straight lines). The parametric equations involve elliptic functions in the first (or "general") case, exponential or rational functions in the various "degenerate" cases. The method of investigation is analytical throughout. Some possible generalizations are outlined at the end. *A. Weil* (Bethlehem, Pa.).

Boyce, M. G. Focal cubics associated with four points in a plane. Amer. Math. Monthly **49**, 226–234 (1942). [MF 6510]

Given an ordered four-point A, B, C, D in the real Euclidean plane, the locus of points X such that the angles AXB and CXD are equal modulo π is called the equal-angle locus of the four-point, and the locus of points Y such that $AY/BY = CY/DY$ is called the equal-ratio locus of the four-point. Each of these loci is a focal cubic (a circular cubic which passes through its singular focus), and the two cubics have the same singular focus and intersect at right angles there and at two other real points in the finite plane; their asymptotes are at right angles. The properties of these curves are investigated by means of complex number representation. A number of questions are discussed, including those of determining the four-points which have a given focal cubic as equal-angle locus, of determining the four-points which have a given focal cubic as equal-ratio locus, and of finding the four-points which have one of a pair of focal cubics having the above intersection and orthogonality properties as equal-angle locus and the other as equal-ratio. *W. V. D. Hodge* (Cambridge, England).

Simpson, Harold. A note on two-circuited circular cubics and bicircular quartics. J. London Math. Soc. **17**, 31–33 (1942). [MF 6972]

A bicircular plane quartic curve is the envelope of a circle having its center on a fixed conic and meeting a fixed circle orthogonally. When the fixed conic is a parabola, the envelope is a circular cubic. In the present note the rectangular Cartesian coordinates are so chosen that the coordinates of the foci and singular foci and the equations of the directrices of the envelopes assume simple forms when they are bipartite. The methods apply to the unipartite case, but less elegantly. *V. Snyder* (Ithaca, N. Y.).

En-Po, Li. Die 28 Doppeltangenten einer Kurve vierter Ordnung. Math. Ann. **118**, 94–111 (1941). [MF 6312]

An elementary and rather ingenious derivation of the numerous properties of the 28 double tangents of a quartic

curve of genus 3 [mostly due to Steiner]. The proofs make use of the Restsatz for linear series on the quartic and also of Hesse's representation of the quartic as the discriminant of a net of quadrics in 3-space. *O. Zariski.*

Hameed, Asghar. On the application of elliptic functions to twisted quartic. Bull. Calcutta Math. Soc. **33**, 9–19 (1941). [MF 6165]

Applying the theory of elliptic functions in the usual way, the author proves a theorem concerning two sets of five points each on a plane cubic, and the corresponding result for a twisted quartic. Then he shows a connection between the set of eight associated points in 3-space and the set of nine associated points in a plane, and he investigates the system of stationary points and a few other groups of points on a twisted quartic. *E. Helly* (Long Branch, N. J.).

Hameed, Asghar. On the projection of the generators of a quadric. Bull. Calcutta Math. Soc. **33**, 21–38 (1941). [MF 6166]

The author examines in a pure geometric way the envelope-conics of the projections of the generators of a quadric from a given point on a given plane. If the vertex of projection varies in space, a system of ∞^3 conics is obtained, which have double contact with the intersection of the given quadric and the plane of projection. The projections of a pencil of quadrics from a given point are a system of conics, each of which touches a plane quartic with two double points at four points. Similarly a ∞^2 system of conics is found, each of which touches a plane cubic with one double point at three points. A few other systems of envelope-conics of certain families of quadrics are investigated, for instance, all quadrics passing through a given conic. *E. Helly* (Long Branch, N. J.).

Apéry, R. Sur les courbes d'ordre n ayant un point multiple O d'ordre $n-4$ et $n-2$ tacnodes, les tangentes tacnodales passant par O. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 552–557 (1941). [MF 6906]

Given a plane algebraic quartic curve of order n with an $(n-4)$-fold point at $(0, 1, 0)$. The curve is expressed in nonhomogeneous Cartesian coordinates by an equation quartic in y, the coefficients being polynomials in x. The condition is then imposed that the curve shall also have a tacnode at $(0, 0, 1)$, with $x=0$ as tacnodal tangent. By means of the invariants of the binary quartic in y it is shown that such curves exist having $n-2$ tacnodal tangents all passing through the multiple point. *V. Snyder.*

Bronowski, J. On triple planes. II. J. London Math. Soc. **17**, 24–31 (1942). [MF 6971]

[Part I appeared in the same J. **12**, 212–216 (1937).] Necessary conditions are obtained for an m-fold plane M to branch along a curve of order $2b$ with $3h$ cusps and $4d$ nodes. These are

$$2b^2(m-2)/m \leqq 3h+4d \leqq 2b^2(m-2)/(m-1).$$

Also, the arithmetic genus of the plane must satisfy

$$p_a \geqq 6b^2(5-m)/(m-1)-3b/2+m-1.$$

These inequalities are obtained by considering the intersections of certain types of curves on a simple model of M, and using the formulas developed in part I.

R. J. Walker (Ithaca, N. Y.).

Ott, E. R. Rational curves defined by an algebraic correspondence. Amer. Math. Monthly **49**, 89–95 (1942). [MF 6240]

(1) Any rational plane envelope can be defined by the joins of corresponding points on two lines in an (m, n) correspondence defined by two involutions. (2) Any plane algebraic curve may be regarded as the envelope of the Pascal line of a hexagon inscribed in a fixed conic when four vertices of the hexagon remain fixed and the other two belong to an algebraic correspondence on the conic.

J. A. Todd (Cambridge, England).

Rangachariar, V. On the conicoids of a pencil touching a given plane. Math. Student **9**, 158–160 (1941). [MF 7032]

There are three quadrics in a pencil touching a given plane. The object of this paper is the discussion of the nature of these three quadrics. It is found that one of them is always real and a hyperboloid of one sheet, and that the nature of the remaining two quadrics depends on the position of the given plane. *E. Helly* (Long Branch, N. J.).

Vest, M. L. A birational T_{n+2} associated with a rational space C_n. Nat. Math. Mag. **16**, 377–380 (1942). [MF 6687]

Snyder, Virgil. Cremona involutions belonging to the Bordiga surface in [4]. Univ. Nac. Tucumán. Revista A. **2**, 203–210 (1941). [MF 6759]

The author defines a particular involutorial Cremona transformation in [3] by means of a family of cubic curves and a pencil of quadric surfaces, and determines its characteristics. The construction is then generalized to [4], the cubic curves being replaced by Bordiga sextic surfaces. The characteristics are partially determined. Finally the extension to $[n]$ is indicated. *R. J. Walker.*

Gauthier, Luc. Une involution d'ordre deux représentant la variété cubique de l'espace à quatre dimensions. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 569–578 (1941). [MF 6908]

This is the third part in a series; the first two papers appeared in the same Bull. (5) **25**, 375–381, 518–529 (1939). Cf. these Rev. **2**, 14.

Godeaux, Lucien. Étude de quelques involutions cycliques appartenant à des surfaces algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 9–21 (1941). [MF 6900]

Godeaux, Lucien. Sur les surfaces algébriques intersections complètes d'hypersurfaces. Univ. Nac. Tucumán. Revista A. **2**, 211–216 (1941). [MF 6760]

An algebraic surface normal in r-space $[r]$, having the property that its canonical system is formed by sections by primes, exists. Among those found are complete intersections. The following theorem includes all algebraic surfaces which satisfy both conditions. A surface F of $[r]$, complete intersection of $r-2$ irreducible algebraic general primals of orders $n_1, n_2, \cdots, n_{r-2}$, is regular and its complete canonical system is cut by primals of order $\sum n_i-(r+1)$. In order that these be primes and F be in $[r]$, each n_i must be a positive integer greater than one and $\sum n_i=r+2$. *V. Snyder.*

Godeaux, Lucien. Sur une propriété de la variété des cordes d'une surface de Veronese. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 411–417 (1941). [MF 6893]

Let y_1, y_2, y_3 be homogeneous coordinates of a point x in a plane, and z_1, z_2, z_3 homogeneous coordinates of a point

in another plane. Let us consider $x_{ik}=y_iz_k$ as homogeneous coordinates of a point in a space of eight dimensions. The equations, obtained by setting the characteristic of the determinant of the x_{ik} equal to one, are the equations of a manifold V_4^6 studied by Segre. If we suppose that $t_{ik}=t_{ki}=x_{ik}+x_{ki}$, $i, k=1, 2, 3$, are homogeneous coordinates in a space of five dimensions, the equation "determinant of $t_{ik}=0$" is the equation of a manifold M_4^3 (the double points of which determine a Veronese surface). The author studies the intersection of M_4^3 with a cubic hypersurface (determined by an equation of third degree in the t_{ik}) and the corresponding canonical surfaces and finds that the genera of its hyperplane sections are $p_a=p_g=5$, $p^{(1)}=10$. Some properties of the intersection of M_4^3 with the hypersurfaces of order $n\geqq 3$ are also found. *G. Fubini.*

Burniat, Pol. Sur les surfaces de bigenre 1 normales dans S_4. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 558–568 (1941). [MF 6907]

It is shown that there can be just two types of normal surfaces in S_4 of bigenus one and having proper multiple curves. One of these has two skew double lines, and the other has four concurrent double lines. That both types exist was shown by the author in an earlier paper [Mém. Soc. Roy. Sci. Liége (4) 1, no. 1, 183–188 (1936)]. *R. J. Walker* (Ithaca, N. Y.).

Longhi, Ambrogio. La determinazione degli ombelichi di una superficie algebrica e alcune questioni piu generali. Univ. Nac. Tucumán. Revista A. **2**, 31–49 (1941). [MF 6747]

In 1876 Voss found that a nonsingular irreducible algebraic surface of order n has $2n(5n^2-14n+11)$ umbilics. An umbilic may be defined as the point P of a surface ϕ such that the two inflectional tangents to ϕ at P meet the absolute. In 1924 Brusotti generalized this, requiring only that the two tangents lie in the same given complex. The author extends the results of Brusotti to singular surfaces and specializes the results for umbilics. He finds that the umbilics of a nonsingular surface of order n occur at the nodes of a curve of order $2n(3n-4)$ and obtains the characteristics of this curve. It should be noted that Salmon [A Treatise on the Analytical Geometry of Three Dimensions, 4th ed., Hodges and Figgis, Dublin, 1882, p. 261, footnote] states that the locus of points of a surface for which the radii of curvature are equal is a curve with double points at the umbilics of the surface. In the case of skew ruled surfaces, the author finds that every ruled surface of order n and genus p which has no special relation to the absolute possesses $6n$ umbilics which lie by threes on $2n$ generators. The characteristics of the curve through the $6n$ umbilics are obtained. *T. R. Hollcroft* (Aurora, N. Y.).

Fano, Gino. Su alcune varietà algebriche a tre dimensioni razionali, e aventi curve-sezioni canoniche. Comment. Math. Helv. **14**, 202–211 (1942).

The three-dimensional varieties M^{2p-2} of order $2p-2$ in linear space of $p+1$ dimensions $[p+1]$ having canonical curve sections of genus p are all rational for $p>10$ with the possible exception of $p=13$, the surface sections of which are reducible to intersections of a general cubic primal in [4] with quadric primals [Fano, Mem. Accad. Naz. Italia 8, 23–64 (1937)]. In the present paper it is shown that the surface sections are complete in the cases $p=9$ and $p=10$, and that the varieties M^{16} and M^{18} are rational. Varieties M^{12} for $p=7$ were shown to be rational in an earlier paper [Fano, Scritti mat. offerti a Luigi Berzolari, Istituto Matematico della R. Università, Pavia, 1936, 329–349], but the present paper gives three representations of M^{12} on [3], one of M^{16} containing systems of lines, one of a M^{16} not containing lines, and one of M^{18}. For $p=13$, the surface intersections are not complete. The only cases still in doubt are those for $p=5$, $p=6$ and $p=8$. *V. Snyder.*

Fano, Gino. Osservazioni sulla rappresentazione di corrispondenze birazionali fra varietà algebriche. Comment. Math. Helv. **14**, 193–201 (1942).

In an earlier paper [Atti Accad. Naz. Lincei. Rend. (6) 11, 329–334 (1930)], the author showed that the canonical curve sections of the three dimensional variety M^{14} of order 14 in [9], of genus 8, the Grassmannian of the lines of [5], is birationally identical with the general cubic primal of [4]. Each of these varieties contain only complete intersection surfaces. In the (1, 1) correspondence between them no isolated fundamental points appear, but each contains a fundamental quintic curve of genus one of the first kind, and 25 fundamental lines of the second kind.

In the present paper the same scheme is generalized to apply to any two three dimensional varieties in (1, 1) correspondence, both of which are complete intersections and have no contact conditions. The prime sections of each consist of multiples of the prime sections of the other, less multiples of base-points and base-curves. Similarly, each element of the base of each system can be expressed by linear combinations of the elements of the other, giving rise to a square matrix with determinant ± 1, analogous to a Cremona transformation between two primes. This analogy is then developed in detail for T_2 in [2], and for T_3 in [3], defined by systems of bilinear equations, including composite base C_6 of the latter. *V. Snyder* (Ithaca, N. Y.).

Turnbull, H. W. The geometry of matrices. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 233–267 (1942). [MF 6937]

Three skew $[k-1]$'s (=medials) in $[2k-1]$ determine a scroll V_k^k consisting of all the lines incident with all three of the medials. After developing some properties of V the author shows that, relative to a given trio of skew medials, a pencil of square matrices of order k can be associated with a unique linear space D of dimension less than k. The canonical invariants of the pencil are then correlated with the geometric relations between V, D and associated loci. *R. J. Walker* (Ithaca, N. Y.).

Hodge, W. V. D. Note on the degeneration of algebraic varieties. Proc. Cambridge Philos. Soc. **38**, 217–219 (1942). [MF 6253]

The paper offers a new and simpler proof of the theorem that an irreducible algebraic variety M of order g and dimension d can be degenerated into a variety consisting of g linear spaces S_d all passing through an S_{d-1}. The original proof by van der Waerden is incomplete without an auxiliary theorem which says that under certain conditions no specialization of M can have a component of dimension less than d. This auxiliary theorem is difficult to prove and the present proof avoids it altogether. *O. Zariski.*

Hodge, W. V. D. The base for algebraic varieties of given dimension on a Grassmannian variety. J. London Math. Soc. **16**, 245–255 (1941). [MF 6619]

On the Grassmannian variety of all S_p's in an S_n there lie the special algebraic subvarieties representing all S_p's

satisfying a given Schubert condition. These subvarieties are of all possible dimensions and are called by the author Schubert varieties. Ehresmann has proved that the Schubert varieties of a given dimension ρ form a base for 2 ρ-cycles of the Riemannian of the Grassmannian variety. In the present note this topological result is strengthened by an algebraic proof to the effect that the Schubert varieties also form an algebraic base in the sense of Severi, that is, that every algebraic subvariety M of the Grassmannian variety belongs to an algebraic system of varieties which contains a variety which is a sum of Schubert varieties. [For subvarieties M of dimension one less than the Grassmannian, this result was first proved by Severi; see Ann. Mat. Pura Appl. (3) 24, 89–120 (1915).] *O. Zariski.*

Hodge, W. V. D. A note on k-connexes. Proc. Cambridge Philos. Soc. 38, 129–143 (1942). [MF 6250]

A k-connex of type $(l_0, l_1, \cdots, l_k)$ is defined to be a form $f(x^0, x^1, \cdots, x^k)$ in $k+1$ sets of indeterminates $(x_0^i, x_1^i, \cdots, x_r^i)$, homogeneous of degree l_i in the indeterminates of the set x^i, and satisfying identically the relations $x_\alpha^i \partial f/\partial x_\alpha^j = 0$, $i<j\leqq k$, where α is a summation suffix. It is assumed that $l_0 \leqq l_1 \leqq \cdots \leqq l_k$, for otherwise f is identically zero. The problem is to find a formula which gives the number of linearly independent k-connexes. This problem is solved in the paper only under the hypothesis that the integers l_i satisfy certain additional conditions [Theorem VII]. It is pointed out, however, that these additional conditions are not satisfied in the interesting special case $l_0=l_1=\cdots=l_k=n$. In this case f is necessarily a form in the determinants which give the coördinates of an S_k in S_r, and thus the k-connex becomes a complex of k-spaces, of degree n. Nevertheless, the formula, although not proved in this special case, is in agreement with the number of linearly independent complexes in cases in which this number has been computed. It is also in agreement with Schubert's formula for the order of the Grassmannian variety. *O. Zariski.*

Zariski, Oscar. Normal varieties and birational correspondences. Bull. Amer. Math. Soc. 48, 402–413 (1942). [MF 6705]

This is a lecture giving a survey of certain developments in the arithmetic theory of algebraic varieties due to the author. It is concerned with the precise nature of a birational correspondence between algebraic varieties, the older and more intuitive ideas not being sufficient for the problems with which the arithmetic theory is largely concerned. Two algebraic varieties V and V', defined over the same field of constants, are birationally equivalent if the field of rational functions on V is isomorphic with the field of rational functions on V'; identifying these fields, a variety W on V corresponds to a variety W' on V' if there exists a valuation of this function field which has W as its center on V and W' as center on V'. "In general," there corresponds to a variety W a unique variety W' of the same dimension, but the exceptional cases are of great importance. These exceptional cases are dealt with by some results of van der Waerden [Math. Ann. 110, 128–160 (1934)] in the case in which both varieties are free from singularities, but when singularities are present van der Waerden's results may break down. Progress is made by considering varieties which are locally normal (which can be characterized geometrically by the property that primals of order n in the ambient space cut on the variety a complete linear system when $n\geqq n_0$) and in particular normal varieties (for which $n_0=1$). The extent to which van der Waerden's theorems can be applied to these varieties is examined.

The paper ends with a study of a special type of transformation, known as a monoidal transformation, which is of importance in the reduction of singularities of an algebraic variety, and the following problems are proposed: (I) Given any birational correspondence between two nonsingular models V and V', and assuming that there are no fundamental points on V', show that the birational transformation can be decomposed into monoidal transformations. (II) Given any birational correspondence between two arbitrary (not necessarily nonsingular) models V and V', and assuming that there are no fundamental points on V', show that the fundamental varieties on V can be eliminated by monoidal transformations. *W. V. D. Hodge.*

Coble, Arthur B. The double-N_n configuration. Duke Math. J. 9, 436–449 (1942). [MF 6881]

The configurations discussed are those on a White surface W_n [Proc. Cambridge Philos. Soc. 22, 1–10 (1923)] in $[n+1]$, consisting of $N_n=(n+2;2)$ lines l_i and N_n spaces $[n-1]\lambda_k$ such that l_i, λ_k are incident when and only when $i\neq k$; W_n is the map of a plane by the linear system of curves of order $n+1$ on a general set P_{N_n} of points $p_1, \cdots, p_{N_n}$ of the plane with grade N_n-1. The pencil of directions about a base point p_i map into points on a line l_i of the configuration under discussion. Let C_j be a curve of order n through all the base points except p_j. These curves are distinct and generic. Each maps into a curve k_j of order N_n-2 in $[n-1]$ which meets each l_i, $i\neq j$. The configuration is formally self-dual. It is descriptively self-dual if for every figure constructed from its parts there exists a dual figure dually constructed from its dual parts. This in turn is intrinsically self-dual if there exists a correlation between l_i and λ_i. For $n=1$, a plane triangle is an N_3. For $n=2$, an example is a double six on a cubic surface.

The present paper shows that, for $n>2$, the generic CW_n is not descriptively self-dual for a generic base $P^2_{N_n}$. Then follow examples of intrinsically self-dual CW_n when the base $P^2_{N_n}$ is formed by the nodes of a rational curve of order $n+3$, generated by the tangents of a rational envelope of class $n+1$ and a perspective conic envelope. The linear system is composed of its adjoints of order $n+1$. Finally the question is raised whether more general symmetric configurations may exist for $n>2$. *V. Snyder.*

Coble, Arthur B. A particular set of ten points in space. Duke Math. J. 9, 450–453 (1942). [MF 6882]

Given a generic trilinear form $T(2, 3, 4)=(\alpha x)(\beta y)(\gamma z)=0$ with digredient variables x, y, z which are cogredient to ξ, η, ζ in their spaces [2], [3], [4]. The system involves 12 absolute constants. There are ten pairs $x, y=p_i, q_i$, neutral for z in $T=0$ [Coble, Duke Math. J. 7, 380–395 (1940); these Rev. 3, 182], connected by a double identity $\sum(q_i\eta)\cdot(p_i\xi)^2=0$. The set P^2_{10} of the ten points p_i in [2] has also 12 absolute constants. The set Q^2_{10} of the ten points q_i is then projectively determined by the identity, hence subject to three projective conditions. It is proved in this paper that these conditions on Q^2_{10} all fall on the tenth point when the first nine are given arbitrarily, in contrast to the conditions on the nodes of a rational plane sextic curve or on the nodes of a symmetroid quartic surface. Various other properties of Q^2_{10} are given. *V. Snyder.*

Emch, Arnold. Zwei spezielle Cremona-Gruppen und die darin auftretenden invarianten Konfigurationen, Kurven und Flächen. Comment. Math. Helv. **14**, 123–133 (1942).

Four general lines in a plane [2] fix the sides and the vertices of a complete quadrilateral V_4. Given the quadratic involution T defined by $x_i'=1/x_i$ in the plane, and the points $(a_1, a_2, a_3)=(a)=P$, $(b)=Q$, and their conjugates P', Q' under T. The lines PQ', $P'Q$ meet in R and PQ, $P'Q'$ meet in R', conjugate of R under T. A web of cubic curves invariant under T is determined. A pencil through the base-points has its remaining intersections at the vertices of a V_4. The twelve nodal cubics of this pencil have their nodes at the four fixed points of T, and at the points L_i, L_i', the L_i being the sides of V_4. Particular cases of this pencil are considered. Properties of the group G_{12}, product of the linear symmetric group of the coordinates and T, is next considered. A certain V_4 is now in six-fold perspective. Curves invariant under this group are of order $3k$, with k-fold points at the base-points of the coordinate system. An outline of the process is extended to [3] by the same transformation. The symmetric linear group and T now generate a G_{48} to which belongs a net of invariant quartic surfaces. [Related papers by the same author are Amer. J. Math. **45**, 192–207 (1923); ibid. **48**, 21–44 (1926).] *V. Snyder.*

Differential Geometry

Santaló, L. A. Some infinitesimal properties of plane curves. Math. Notae **1**, 129–144 (1941). (Spanish) [MF 6281]

In the greater part of his paper the author discusses a set of problems which are connected with, and solved by means of, the canonical representation of a Euclidean plane curve in the neighborhood of one of its points A and which consist in the determination of the limits of the ratios of certain segments and areas. The following example is typical: Let B be a second point of the curve and B' a point on the tangent to the curve at A, both lying on the same side of the normal to the curve at A. The straight line $B'B$ shall meet this normal in P. What is the limit of P if B and B' approach A in such a way that the quotient of their distances from A is a given constant? The last pages deal with analogous problems for affine curves. *P. Scherk.*

DeCicco, John. General comparison of conformal and equilong geometries. Nat. Math. Mag. **16**, 275–279 (1942). [MF 6353]

A brief expository and historical account of two geometries of the plane, the conformal and the equilong, with emphasis on their similarities and distinctions. Several recent results are included. *P. Franklin.*

DeCicco, John. The two conformal covariants of a field. Univ. Nac. Tucumán. Revista A. **2**, 59–66 (1941). [MF 6749]

It is shown that any nonisothermal field of plane lineal elements (family of curves) possesses two conformal differential covariants of third order. If two fields have the same covariants then they are equivalent under the conformal group of transformations of the plane. Two geometric interpretations of these covariants are obtained. One of these interpretations is given by means of "symmetric three webs." The other depends upon associating a two dimensional Riemannian metric with any plane element field. *A. Fialkow* (Brooklyn, N. Y.).

Kasner, Edward and DeCicco, John. Conformal geometry of third order differential elements. Univ. Nac. Tucumán. Revista A. **2**, 51–58 (1941). [MF 6748]

It was shown by Kasner and subsequently by Comenetz that the totality of conformal transformations in the plane induces a continuous six parameter group G_6 operating on the ∞^3 third order differential elements at a fixed point. In this paper a detailed study of the transformations belonging to G_6 which are of period two (called involutions and symmetries) is made. The following theorems are examples of the type of results which are obtained: (1) The group G_6 can always be factored as the product of four or fewer symmetries and a magnification. (2) Any n ($n\geqq 3$) differential third order elements at a point of which no two are tangent possess $3n-6$ differential invariants under G_6 and these invariants form a complete set. (3) Any ∞^1 third order differential elements at a point, one in each direction ("element series"), may be characterized by two differential invariants with respect to G_6. *A. Fialkow.*

Kasner, Edward and DeCicco, John. Pseudo-conformal geometry: functions of two complex variables. Bull. Amer. Math. Soc. **48**, 317–328 (1942). [MF 6413]

Die Verfasser stellen die Differentialinvarianten erster Ordnung auf, die bei analytischen (pseudokonformen) Abbildungen zwischen zwei durch einen gegebenen Punkt laufenden Kurven-, Flächen- oder Hyperflächenelementen bestehen. Es werden die sechs möglichen Fälle untersucht: Im Falle zweier allgemeiner Kurvenelemente, oder zweier allgemeiner Hyperflächenelemente oder im Falle eines Kurven- und eines allgemeinen Flächenelementes, oder eines allgemeinen Flächen- und eines Hyperflächenelementes gibt es keine Invarianten; das heisst, jedes Paar solcher Elemente lässt sich durch eine geeignete pseudokonforme Abbildung in jedes andere Paar vom gleichen Typus überführen. Im Falle eines Kurven- und eines Hyperflächenelementes gibt es eine Invariante [der "Pseudowinkel"; siehe Kasner, Trans. Amer. Math. Soc. **48**, 50–62 (1940); these Rev. **2**, 86] und im Falle zweier allgemeiner Flächenelemente zwei unabhängige Invarianten. Ausnahmen bestehen, wenn, zum Beispiel, zwei Kurvenelemente auf dem gleichen analytischen Flächenelemente liegen (genauer: wenn sie ein "isoclinales" Kurvenpaar sind) und in ähnlichen Fällen. *P. Thullen* (Quito).

Kasner, Edward and DeCicco, John. The general invariant theory of irregular analytic arcs or elements. Trans. Amer. Math. Soc. **51**, 232–254 (1942). [MF 6316]

An analytic element is defined by writing x and y as integral power series (without constant term) in a parameter t. This paper gives a complete classification of analytic elements under the group of arbitrary analytic point transformations. There is one regular type, essentially y an integral power series in x, with canonical form $y=0$. There are 12 irregular types. For each of these types a canonical form is found and the nature of the differential and arithmetic invariants, if there are any, is discussed. *P. Franklin* (Cambridge, Mass.).

Kasner, Edward and DeCicco, John. Generalized transformation theory of isothermal and dual families. Proc. Nat. Acad. Sci. U. S. A. **28**, 52–55 (1942). [MF 6136]

The authors characterize completely those transformations which carry isothermal fields into isothermal fields. The transformations considered are field-element to lineal-element as well as contact transformations. Similar results are found for dual-isothermal fields. *P. Franklin.*

Kasner, Edward and De Cicco, John. Transformation theory of isogonal trajectories of isothermal families. Proc. Nat. Acad. Sci. U. S. A. **28**, 328–333 (1942). [MF 7068]

Using a type of minimal coordinates, the authors determine explicitly the form which a group of curvature-element-transformations in the plane must take if it converts all the isogonal trajectories of any one isothermal family of curves into another such family. *P. Franklin.*

Kasner, Edward. Transformation theory of isothermal families and certain related trajectories. Univ. Nac. Tucumán. Revista A. **2**, 17–24 (1941). [MF 6745]

A lineal element in the plane consists of the triplet of numbers (x, y, θ), where the first two are the Cartesian coordinates of a point and the last is the inclination of a direction through the point. It is shown that the entire group of lineal element transformations which convert all isothermal families of curves into isothermal families is given by the equations $X=\phi(x, y)$, $Y=\psi(x, y)$, $\Theta=a\theta+h(x, y)$, where ϕ and ψ satisfy the Cauchy-Riemann equations, h is any harmonic function of (x, y) and a is a nonzero constant. As a corollary of this theorem it follows that conformal transformations are the only point transformations of the plane which carry all isothermal families into isothermal families.

The paper also considers the geometry connected with the element transformations $X=x$, $Y=y$, $\Theta=a\theta+b$. If all these transformations are applied to a given field of plane elements, a set of ∞^3 integral curves is obtained. This family of curves enjoys the geometric property I discovered by the author in his work on dynamical trajectories [Differential-geometric Aspects of Dynamics, Amer. Math. Soc. Colloquium Publ., vol. 3, part II, New York, 1913 (reprinted 1934)]. *A. Fialkow* (Brooklyn, N. Y.).

Kasner, Edward and Mittleman, Don. A general theorem on the initial curvature of dynamical trajectories. Proc. Nat. Acad. Sci. U. S. A. **28**, 48–52 (1942). [MF 6135]

The theorem is: If a particle starts from "maximum rest" in an acceleration field of order n, the initial curvature of the trajectory is $n!(n-1)!/(2n-1)!$ times the curvature of the line of force through the initial position. The special case $n=2$ was formulated by Kasner in 1912.

P. Franklin (Cambridge, Mass.).

Fialkow, Aaron. The conformal theory of curves. Trans. Amer. Math. Soc. **51**, 435–501 (1942). [MF 6550]

The author develops the conformal geometry of a single curve in a Riemann space V_n of $n>2$ dimensions. For $n=2$, the circular conformal geometry is studied since there is no two dimensional theory of a single analytic curve. Considerable use is made of the conformal derivative of a conformal tensor with respect to a conformal parameter along a given curve. This new type of tensor differentiation, which is dependent upon the curve, plays here a role analogous to that of covariant differentiation in the metric theory. The conformetric tensors, which have both metric and conformal properties, are used very widely. Thus, for a conformetric vector λ^i, both its direction and metric length are invariant. A unit conformetric vector η^i and a conformal arc length S are found in terms of the metric unit normals and metric curvatures, so that a systematic use of the conformal derivative with respect to S, beginning with the derivative of η^i, yields a system of Frenet equations. In contradistinction to the metric theory, η^i is not the unit tangent vector ν^i, since the conformal derivative of ν^i is zero. The Frenet process defines $(n-1)$ conformal unit normals and $(n-2)$ conformal curvatures $J_1, J_2, \cdots, J_{n-2}$.

By an entirely different process, another conformal curvature J_{n-1} is obtained. This is independent of the preceding ones. Let V_n be mapped conformally into $\bar{V}_n$ so that the curve C of V_n is carried into the curve $\bar{C}$ of $\bar{V}_n$. Then the conformal curvatures $J_1, J_2, \cdots, J_{n-1}$ are the same functions of S for both C and $\bar{C}$. Conversely, if V_n and $\bar{V}_n$ are conformal to a Euclidean space R_n, and if the J's are the same functions of S for both C and $\bar{C}$, then a conformal map of V_n onto $\bar{V}_n$ exists so that C and $\bar{C}$ correspond. The existence theorem is that, in any V_n, a curve C exists for which the J's are preassigned continuous functions of S. This curve C is uniquely determined by a preassigned set of initial conditions. The most general conformal differential invariant of a curve in a V_n conformal to an R_n is a function of the J's and their derivatives with respect to S. For $n=2$, only those V_2's are considered which admit circular conformal transformations. A circular conformal arc length S is defined, and a new circular conformal curvature J_1 is obtained analogous to J_{n-1}. All the preceding results are now valid for a V_2 if the word "conformal" is replaced by "circular conformal."

Other topics considered are conformal null curves, curves in an Einstein space, curves in a conformal Euclidean space $\bar{R}$, groups of conformal transformations in an R_n and $\bar{R}_n$, circular conformal transformations and curves in a conformal space. *J. DeCicco* (Chicago, Ill.).

Kerawala, S. M. On "quasi-helices" associated with curves. Proc. Benares Math. Soc. (N.S.) **2**, 91–100 (1940). [MF 6653]

Let P be a point of a given space curve C. In the normal plane of C at P a circle of radius a with center in P is described, and on this circle a point P' of a curve C' is chosen such that the ratio ds'/ds of the arc elements of the two curves is a given constant. The author calls C' a "quasi-helix" associated with C. He gives the differential equation of C', computes its curvature and torsion and examines a few particular cases. *E. Helly* (Long Branch, N. J.).

Bell, P. O. The parametric osculating quadrics of a family of curves on a surface. Amer. J. Math. **64**, 343–347 (1942). [MF 6436]

At points P, P_1, P_2 of an asymptotic curve of a surface S construct the tangents to the curves of a one parameter family of curves on S. If the quadric determined by these three tangents has a limit as P_1 and P_2 independently approach P along the asymptotic, this limit is called a parametric osculating quadric. Evidently there are two such quadrics for each one parameter family of curves on S. The curve of the family through P is denoted by C_λ.

Certain properties of these quadrics are deduced. Among these may be noted the following: At a general point on the tangent to C_λ at P, the tangent planes to the two parametric osculating quadrics separate harmonically the oscu-

lating plane of C_λ at P and the tangent plane of S at P. Let l be a line in the tangent plane to S at P, and l' be the reciprocal polar of l with respect to the quadric of Lie. The line l' generates a congruence Γ'. The reciprocal polars of l with respect to the two parametric osculating quadrics are coplanar with the conjugate of the tangent to C_λ if and only if the given one parameter family of curves are union curves of the congruence Γ'.

The following characterization of the curves of Darboux is given: The complete intersection of the quadric of Lie and a parametric osculating quadric is two straight lines. These lines determine a plane. A similar statement holds for the second parametric osculating quadric. The curve C_λ is a curve of Darboux if and only if the harmonic conjugate of the tangent plane of S at P with respect to these two planes contains the axis of the conjugate net determined by the given family of curves. *V. G. Grove.*

Chenkuo Pa. On the quadrics of Moutard. Univ. Nac. Tucumán. Revista A. 2, 67–77 (1941). [MF 6750]

The locus of the osculating conic to the section of a surface S produced by a plane passing through a nonasymptotic tangent at a generic point O of S is a Moutard quadric having contact of the second order with S at O. The author's purpose is to find residual osculating conics on a given Moutard quadric and to derive from these a characteristic property of a Moutard pencil of quadrics. It is shown that: "At any point O of any surface except the surface of coincidence there are three osculating conics c_1, c_2, c_3, through each of which pass three Moutard quadrics." Another result is the following. Through every nonasymptotic tangent of the surface at O there exist two plane sections whose osculating conics lie on two distinct Moutard quadrics. The harmonic conjugate of the tangent plane of the surface at O with respect to these two planes envelopes a cone of class 6 when the tangent varies. This cone is used to furnish simple definitions for many canonical lines. *N. Coburn.*

Godeaux, Lucien. Note sur les surfaces dont les quadriques de Lie n'ont que trois points caractéristiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 430–437 (1941). [MF 6895]

Let S be a surface. The author supposes that the Lie quadric corresponding to any point x of S has only three characteristic points: the point x and two points x_i $(i=1, 2)$ which generate two surfaces S_i. He considers the congruences of the lines $r_i=xx_i$ and of the lines s_i of intersection of the planes tangent to S at x and to S_i at x_i. The focal points and focal planes of these congruences are determined, and some of their properties stated. For example: Two corresponding asymptotic tangents to S, S_i at corresponding points contain the same focal point of the corresponding line s_i. The intersection of a focal plane of a line s_i with the corresponding line r_i, and the corresponding focal point of r_i are harmonic conjugates with respect to the points x, x_i. *G. Fubini* (New York, N. Y.).

Godeaux, Lucien. Note sur les surfaces dont les quadriques de Lie ont cinq points caractéristiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 487–498 (1941). [MF 6898]

Each point x of a surface S determines a corresponding Lie quadric. We suppose that this quadric has five characteristic points: the point x and four other points x_{ik} $(i, k=1, 2)$. Each point x_{ik} generates a surface S_{ik}. Furthermore, there is associated with x_{ik} the line r_{ik} joining x with x_{ik} and the line s_{ik} of intersection of the planes tangent to S at x and to S_{ik} at x_{ik}. The author determines the developable surfaces, focal points and planes of the congruences generated by the lines r_{ik} and by the lines s_{ik}. He supposes first that the asymptotic lines of S correspond to the asymptotic lines of each of the four surfaces S_{ik}, and proves, among other things, that every focal plane of r_{ik} contains one of the focal points of the corresponding line s_{ik}. Then other special surfaces S are studied, for which the asymptotic lines correspond to the developable surfaces of the congruence generated by the lines r_{11} (and consequently also to the developable surfaces of the congruences r_{22}, s_{12}, s_{21}). *G. Fubini* (New York, N. Y.).

Fubini, G. On a class of surfaces. Proc. Nat. Acad. Sci. U. S. A. 28, 178–185 (1942). [MF 6606]

Analytic conditions that a surface S possess at least two systems of curves such that the curves of each system are projectively equivalent. The author finds all surfaces S for the particular case in which S is not an algebraic surface and the trajectories of the projectivities involved are the curves of the two systems. *V. G. Grove.*

Beck, H. Kurvenkongruenzen und volumentreue Transformationen. Math. Z. 47, 275–290 (1941). [MF 6823]

Linear congruences are first classified by means of the rank of the second quadratic form of the congruence and the rank of the matrix (H, H_u, H_v), where H is the direction vector field of the congruence. It is shown that every analytic congruence remains linewise invariant under a Lie group of volume-preserving transformations and also under at least one volume-preserving involution (other than the identity). It is proved that every analytic curve and surface remains pointwise invariant under a group of volume-preserving involutions. The last section gives three types of area-preserving transformations that were missed by Scheffers in his classification (due to specializing the value of one constant). *M. Knebelman* (Pullman, Wash.).

Finikoff, S. Congruences associées dans une déformation simultanée. Bull. Soc. Math. France 68, 53–82 (1940). [MF 6808]

According to Cartan two figures generated by elements M and N are said to be applicable by a deformation of the nth order if there exists a one-to-one correspondence between M and N, and if each pair of corresponding elements and their corresponding neighborhoods of the nth and lower orders may be also made to coincide by one and the same projectivity. The present paper considers the simultaneous deformation of two congruences. In particular it is found that, if a congruence Γ is given, one may associate with Γ by means of three arbitrary functions of two parameters a congruence Γ' such that Γ and Γ' are applicable by a deformation of the first order. It is found that, if Γ and Γ' are applicable by a deformation of the first order, then their developables correspond, and the projectivity appearing in the definition makes the corresponding foci and focal planes coincide.

Two congruences are said to be related in a transformation T if the lines joining the corresponding foci of the lines of the congruence are tangent to the two focal surfaces, that is, the skew quadrilateral formed by the lines joining the foci of the congruences may be arranged so that each consecutive pair generates congruences having a focal surface in common. Two congruences related by a transforma-

tion T preserve that property after a simultaneous deformation. Several particular cases of deformations of this type are discussed: W congruences with ruled focal surfaces, pairs of congruences reciprocal with respect to the null system of a linear complex, periodic (of period four) sequences of Laplace, and finally congruences which are stratifiable (that is, the lines of each congruence are the intersection of the focal planes of the other).

V. G. Grove (East Lansing, Mich.).

Dobbrack, Gerhard. Differentialgeometrie der Kugelkomplexe. II. Minimalkomplexe. Math. Z. 47, 1–7 (1940). [MF 6292]

Sphere complexes for which the first variation of a certain invariant integral vanishes are called minimal complexes (in analogy to minimal surfaces). They are characterized by the vanishing of an invariant H which is the analogue of mean curvature of surfaces. Some geometrical properties of minimal complexes are derived. *H. Samelson.*

da Cunha, Pedro José. Du parallélisme dans l'espace euclidéen. Portugaliae Math. 2, 177–246 (1941). [MF 5829]

This paper gathers together the results of numerous articles presented by the author to Spanish and Portuguese academies since 1932 concerning the properties of (1) parallel curves, both plane and skew, (2) parallel surfaces, (3) parallelism of a curve and a surface, and presents them in extended form. The definitions adopted by the author are (1) a plane curve C' is parallel to a plane curve C provided C' is the envelope of a one-parameter family of circles with constant radius and centers at the points of C; while the parallels of a skew curve C are the intersections of developable surfaces whose generators are normals of C, with the envelope of a one-parameter family of spheres with constant radius and centers the points of C. (2) The surface S' is parallel to S provided S' is the envelope of a two-parameter family of spheres of constant radius and centers the points of S. (3) A curve C is parallel to a surface S whenever the normals to S through the points of C all have the same length segment cut off by C and S. Detailed derivations of curves parallel to the parabola, cycloid and logarithm curves are given. The treatment is entirely elementary, based on Cartesian coordinate representation of the curves and surfaces and uses only the simplest concepts of differential geometry. The only difficulties which are encountered relate to the solutions of certain ordinary differential equations. *L. M. Blumenthal.*

Bouligand, Georges. Sur les surfaces satisfaisant à la théorie de Gauss et à la relation classique entre la courbure totale et la courbure géodésique. J. Math. Pures Appl. (9) 20, 325–337 (1941). [MF 6837]

It is shown that the formula of Gauss-Bonnet applies to any surface S with the following properties: every point of S has a neighborhood which can be represented in the form $z=f(x, y)$, where $f(x, y)$ has continuous second derivatives. Let a line element ds^2 be given whose coefficients E, F, G are analytic functions of u, v. The differential equations for the coordinates $x(u, v)$, $y(u, v)$, $z(u, v)$ of a surface with the given ds^2 are of the second order, but in their derivation Darboux uses equations of the third order. Darboux's procedure is modified so that only equations of the second order are used. *H. Busemann* (Chicago, Ill.).

Chern, Shiing-shen. The geometry of isotropic surfaces. Ann. of Math. (2) 43, 545–559 (1942). [MF 7009]

This paper studies the geometry of a general three-space in which there is given a two-parameter family of surfaces. The crux of the investigation is the construction of a unique Weyl geometry whose elements are the contact elements of the given surfaces. The method is that used extensively by E. Cartan in similar investigations; it consists in associating a general reference frame and infinitesimal displacement with each element of the space and imposing invariant geometrical conditions until the frame and displacement are uniquely determined. In the present case the space of elements is four-dimensional while the frames are three-dimensional. It is then shown that the question of equivalence of two two-parameter surface families reduces to the equivalence problem for the associated Weyl geometries. This is treated, following Cartan, as a problem in the equivalence of Pfaffian systems. As by-products a process of covariant differentiation and a complete system of invariants arise. Finally, the condition that the geometry be a Weyl point geometry is expressed by the vanishing of certain invariants, one of which is given a geometrical interpretation. According to an addendum the author has also succeeded in solving the general problem of an $(n-1)$-parameter family of hypersurfaces in n-space.

J. L. Vanderslice (College Park, Md.).

Beckenbach, E. F. The analytic prolongation of a minimal surface. Duke Math. J. 9, 109–111 (1942). [MF 6344]

In classical differential geometry we have the following theorem. If a minimal surface S cuts a plane orthogonally, then S is symmetric with respect to the plane. Known proofs depend essentially upon the assumption that the intersection of S and the plane is an interior curve on S. As a generalization the author establishes the following theorem. If a minimal surface S is bounded in part by an arc C of a curve that lies in a plane π, and if S approaches π orthogonally, then S can be continued analytically as a minimal surface across π, and the extended surface is symmetric with respect to π. The proof is based upon the conformal representation of a minimal surface in terms of harmonic functions and on theorems concerning the analytic continuation of harmonic functions. The paper concludes with a remark on a simplification of an argument used by Douglas in proving a related theorem on minimal surfaces. *T. Radó* (Columbus, Ohio).

Morse, Marston and Hedlund, Gustav A. Manifolds without conjugate points. Trans. Amer. Math. Soc. 51, 362–386 (1942). [MF 6320]

Of all the closed two-dimensional manifolds, only the torus and the Klein bottle can have a flat metric (one for which the Pythagorean theorem is valid for sufficiently small geodesic triangles); this is a known corollary of the Gauss-Bonnet theorem. The first part of this paper seeks, in terms of conjugate points and focal points of the geodesics on these surfaces, sufficient conditions to guarantee flatness. The condition that no geodesic has a pair of conjugate points (a condition fulfilled by flat manifolds) seems almost to suffice, but the authors are able to prove flatness only under a more restrictive condition, a condition which is guaranteed if no geodesic has focal points (points analogous to centers of curvature). Using only the condition on conjugate points the authors prove that any geodesic when extended indefinitely in both directions either is a simple

arc or covers a simple closed curve repeatedly. The question of what other properties, if any, are necessary to guarantee flatness is not answered.

The second part of the paper concerns surfaces of higher genus. In this part the authors prove the existence of transitive geodesics on orientable surfaces of genus greater than one and nonorientable of genus greater than two (surfaces of more complicated structure than the torus and the Klein bottle, respectively) under the condition that no geodesic on the surface has a pair of conjugate points. Earlier proofs of the existence of transitive geodesics have been based upon assumptions of everywhere negative curvature or instability of various sorts, assumptions which guarantee absence of conjugate points.

In the paper essential references are made to half a dozen earlier papers by the authors and others.

C. B. Tompkins (Washington, D. C.).

Hamburger, Hans Ludwig. Beweis einer Caratheodoryschen Vermutung. II. Acta Math. **73**, 175–228 (1941). [MF 6100]

Hamburger, Hans Ludwig. Beweis einer Caratheodoryschen Vermutung. III. Acta Math. **73**, 229–332 (1941). [MF 6101]

In order to prove Carathéodory's conjecture that on every closed (analytic) surface of genus 0 there are at least two umbilical points one has to show that the index of an umbilical point with respect to the lines of curvature is always nonnegative. In the first part [Ann. of Math. (2) **41**, 63–86 (1940); cf. these Rev. **1**, 172] the author has established this estimate under the assumption of a certain "regularity" condition for the surface. In the present two papers he gives the proof for the general case by a very long and detailed study of the differential equation in question.

H. Samelson (Princeton, N. J.).

Barankin, E. W. Heat flow and non-Euclidean geometry. Amer. Math. Monthly **49**, 4–14 (1942). [MF 6140]

Similar to an example, given by Poincaré for the 3-space, the author constructs a 2-dimensional Lobatschevskian geometry in the following way. In a two-dimensional plane, let the temperature be given by the following function of the Euclidean coordinates: $T=b(y-1/pb)$, where p and b are constants. Distances are measured with an infinitesimal rod, whose thermal coefficient of expansion is supposed to be p for all T, and whose length is taken as constant in the geometry to be constructed. The line element in this metric is given as $ds^2=(dx^2+dy^2)/(pb)^2y^2$. The author develops the fundamental formulas of this geometry, shows that a three parametric group of congruence transformations exists, and that the geodesics with respect to the impressed metric are the upper half-circumferences of the circles which are centered on the x-axis. The cross ratio formulas for distance and angle are derived, and the identity with the Lobatschevsky geometry is easily established.

E. Helly (Long Branch, N. J.).

Rosenson, N. Sur les espaces Riemanniens de classe I. II. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 325–351 (1941). (Russian. French summary) [MF 5839]

[Part I appeared in the same Bull. **4**, 181–192 (1940); cf. these Rev. **2**, 163.] Here we find explicit expressions for the quantities p_s for odd s, supplementing those given for even s in the previous article. These are obtained by expressing $-(s+1)(s-2)p_{s-1}p_{s+1}+s(s-1)p_s^2$ in terms of the functions U and using the formulas for even s. Then follows the discussion of Riemannian manifolds of class I with special properties, for instance, with the special condition fulfilled:

$$^{(2)}W_{\alpha\beta}^{\cdots\gamma\delta}=\frac{^{(2)}W_1}{n(1-n)}(\delta_\alpha{}^\gamma\delta_\beta{}^\delta-\delta_\beta{}^\gamma\delta_\alpha{}^\delta).$$

The paper ends with a chapter in which necessary and sufficient conditions are established that a Riemannian manifold be of class I. The proofs are given for the 5 cases: $p_3\neq 0$; $p_3=0$, $p_5\neq 0$; $p_3=0$, $p_5=0$, $p_1p_4\neq 0$; $p_3=p_5=p_1p_4=0$, $p_4\neq 0$; and $p_3=p_4=0$. In the general case these equations are of degree four in the Riemann-Christoffel tensor, but there are cases in which this degree may be six. It would be interesting to establish the relations between Rosenson's formulas for p_s and the expressions in the older literature [Lipschitz, Killing, etc.; see Struik, Grundzüge der mehrdimensionalen Differentialgeometrie, Springer, Berlin, 1922, p. 133].

D. J. Struik (Cambridge, Mass.).

Thomas, T. Y. Families of Riemannian spaces. Revista Ci., Lima **43**, 319–353, 455–464, 631–650 (1941); **44**, 27–44 (1942). [MF 5738]

The author considers infinite families of Riemannian spaces which are defined over an underlying coordinate manifold M, where M is connected, compact and separable of class C^{r+1}. It is shown that, in the set T of all Riemannian metrics of M, a distance function can be defined. Hence the set T becomes a metric space $T_1(M)$. The author shows that $T_1(M)$ is complete and discusses other properties of this metric space. Thus it is shown that: "Let M be a manifold of class C^{r+1} with $r\geqq 1$ and let $T_1(M)$ be the complete metric space whose elements are the Riemannian metrics of class C^r defined over M. Then, if F is any infinite bounded set of points in $T_1(M)$, it is possible to select from the set F a sequence $h^{(1)}$, $h^{(2)}$, $\cdots$ which converges to a limit q in the space $T_1(M^*)$."

In the second section the author studies the automorphisms of the manifold M with the aid of an invariant distance function ρ and a series of inequalities on quantities ψ^α (which denote a (1, 1) continuous point transformation of M into itself). A typical result is: "Let G be an infinite family of automorphisms of manifold M such that the inequality (e_1) is satisfied for every automorphism ψ of G. Then, given any positive number ϵ, we can find a positive number η such that $\rho(\psi(P), \psi(Q))<\epsilon$ whenever $\rho(P, Q)<\eta$, for every automorphism ψ of G." Next the author defines the notion of the derived Riemannian space p' and introduces a new distance function in T. It is shown that the set T becomes now a semi-metric Hausdorff space $T_2(M)$. Properties of the derived spaces p^*, formed on $T_2(M)$, are discussed. Thus it is shown that, if the given space p is of class C^r, so also is the derived space p^*. Moreover, the fundamental metric tensor of the space p^* does not vanish at any point of the manifold M and the covariant derivative of this tensor vanishes over M.

Finally the author defines the notion of a topologically nondecomposable manifold M. With the aid of the theory of derived spaces, he proves: "If the manifold M is topologically nondecomposable, the set of elements in $T_2(M)$ which represent Riemann-Einstein spaces is an isolated set."

N. Coburn (Austin, Tex.).

Mikami, Misao. On the theory of non-linear direction displacements. Jap. J. Math. **17**, 541–568 (1941). [MF 5647]

This paper extends the work of H. Friesecke and E. Bortolotti on the theory of nonlinear displacements. First, the author introduces a direction element, the vector ξ^k. The covariant differential of this direction element and of a vector function $v^k(x, \xi)$ of the direction element ξ^k and the space coordinates x^k are then defined by

$$\delta\xi^i = d\xi^i + \Gamma^i_j(x, \xi)dx^j, \tag{1}$$

$$\delta v^i = dv^i + \Gamma^i_{jk}(x, \xi)v^k dx^j, \tag{2}$$

where Γ^i_j is nonlinear but homogeneous of order one in the ξ^k and

$$\Gamma^i_{jk}(x, \xi) = \partial\Gamma^i_j/\partial\xi^k. \tag{3}$$

Various curvature affinors are constructed. It is shown that, if the direction element field is parallel, then the associated vector field is parallel. Finally, the author studies two types of vector displacements: (1) those which furnish the same direction displacement; (2) those which form a generalized parallel field. In the first case, the argument leads to a preferred displacement; in the second case, a necessary and sufficient condition for this generalized parallelism is obtained. A similar study is made for the cases in which the direction element is replaced by (1) a bivector, (2) a p-vector, (3) a symmetric tensor. *N. Coburn.*

Suguri, Tuneo. Projective theory of generalized affine connections. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **2**, 91–106 (1941). [MF 5639]

By a process of specialization an affine connection is obtained having components depending on the coordinates x^α and the directions dx^α/dt. Parabolas in a generalized sense are introduced by means of a correspondence involving parabolas in the tangent spaces. Several theorems dealing with the order of contact between such parabolas are given. The affine representation of the projective theory is then treated by the method of Thomas and certain analogues of well-known invariants obtained.

T. Y. Thomas (Los Angeles, Calif.).

Hombu, Hitoshi and Suguri, Tuneo. A treatment of geometric quantities in the manifold of surface-elements. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **2**, 67–90 (1941). [MF 5638]

This paper generalizes the work of J. Douglas on the general geometry of paths and of systems of k-dimensional manifolds in an n-dimensional coordinate space S_n [Ann. of Math. (2) **29**, 143–168 (1928); Math. Ann. **105**, 707–733 (1931)]. The papers of Douglas investigate the simplest case in which the k-dimensional surface elements are defined by a system of partial differential equations of the second order. In the present work, each surface element is defined by the set of quantities which includes: (a) the point coordinates x^i $(i=1, 2, \cdots, n)$ of the space S_n, (b) the parameters u_j $(j=1, 2, \cdots, k)$ of the k-dimensional surface element, (c) the partial derivatives (up to an arbitrary order ν) of the coordinates x^i with respect to the parameters u_j. The set of these quantities are the components of an analytic surface element of order ν. The geometry of these surface elements is developed by a consideration of the group $G_{(\nu)}$ of transformations between the corresponding components of the various analytic surface elements which describe the same geometric surface element in different coordinate and parameter systems. The infinitesimal operators of $G_{(\nu)}$ and of an extended group associated with $G_{(\nu)}$ are calculated and various formal results are derived.

A. Fialkow (Brooklyn, N. Y.).

Ohkubo, Takeo. A symmetric connection in an n-dimensional Kawaguchi space. Proc. Imp. Acad. Tokyo **17**, 178–181 (1941). [MF 5398]

Let K_n be an n-dimensional space in which the arc length s of a curve $x^i = x^i(t)$ is given by

$$s = \int\{A_i(x, x')x''^i + B(x, x')\}^{1/\nu}dt.$$

Naturally, the A_i and B are such that the arc length remains unaltered by any change in the parameter t. The geometry of this space was first developed by Kawaguchi [Trans. Amer. Math. Soc. **44**, 153–167 (1938)]. In this paper, the author indicates an alternative development by means of a symmetric affine connection $\Gamma^i_{jk}(x, x')$. This connection is defined if $p \neq 3/2, 3$ and is unaltered by transformations of the parameter t. The various curvature and torsion tensors formed from the Γ^i_{jk} are calculated and the customary relations which they satisfy are stated. Under special conditions, the author's symmetric connection becomes identical with one already defined by Kawaguchi.

A. Fialkow (Brooklyn, N. Y.).

Tonowoka, Keinosuke. On a metric connection along a curve in a special Kawaguchi space of order two. Proc. Imp. Acad. Tokyo **17**, 182–185 (1941). [MF 5399]

Let K_n be the special Kawaguchi space of order 2 defined in the previous review. Using methods largely due to Kawaguchi, the author shows that it is possible to define a connection and a tensor $g_{ij}(x, x', x'')$ which are unchanged by transformations of the parameter t. The tensor g_{ij} is such that the integral for the arc length may be written as $s = \int\{g_{ij}x'^i x'^j\}^{\frac{1}{2}}dt$, formally similar to the corresponding integral for a Riemann space. For this reason g_{ij} may be considered as the metric tensor of K_n. The results of this paper are valid for every p except for $p=1, 3/2, 5/2, 3$.

A. Fialkow (Brooklyn, N. Y.).

Berwald, L. Ueber Finslersche und Cartansche Geometrie. I. Geometrische Erklärungen der Krümmung und des Hauptskalars eines zweidimensionalen Finslerschen Raumes. Mathematica, Timişoara **17**, 34–58 (1941). [MF 6730]

Let $edx^2 + 2fdxdy + gdy^2$, where e, f, g are functions of x, y, x', y', be the line element of a two-dimensional Finsler space Σ. Consider a field of curves in Σ. The line element of the field determines e, f, g as functions of x and y only, therefore a Riemann metric which is called the osculating metric along the field. The curvature K of this metric depends only on the line element, not on the choice of the field in which it occurs. Besides new derivations for older geometric interpretations for K and the extremal curvature ρ of a curve, new interpretations are obtained. Here the following examples must suffice. Let Γ_0 be a curve through a given line element (x_0, y_0, x_0', y_0'). As field take the curves parallel to Γ_0 (equidistant to Γ_0) and its orthogonal trajectories, and as parameters the arc length s_0 on Γ_0 measured from (x_0, y_0) up to the trajectory and the arc length S on this trajectory. Consider the quadrangle $(0, 0)$, $(s_0, 0)$, (s_0, S), $(0, S)$; call Ω its area in the osculating Riemann metric and s the distance between the last two

vertices measured on the curve Γ parallel to Γ_0. Then

$$K(x_0, y_0; x_0', y_0') = \lim_{\substack{S\to 0 \\ s\to 0}} (s_0^2 - s^2)/\Omega^2.$$

If Γ_0 is an extremal we have $K = \lim_{S\to 0} 1/S\cdot\rho$, where ρ is the extremal curvature of Γ at $(0, S)$. Let r, ϕ be Kneser's polar coordinates with a given origin (x_0, y_0) (that is, ϕ is the Landsberg angle). Let Ω be the area of the sector $0\leqq r\leqq R$, $0\leqq\phi\leqq\Phi$ and σ the length of the arc $r=R$, $0\leqq\phi\leqq\Phi$, both measured in the Riemann metric osculating along the field $\phi=$const. Then

$$K(x_0, y_0; 1, 0) = 6\lim_{\substack{R\to 0 \\ \phi\to 0}} R\phi - \sigma/R^3\phi = 24\lim_{\substack{R\to 0 \\ \phi\to 0}} ((R^2/2)\phi - \Omega)/R^4\phi.$$

If I designates the principle scalar (Hauptskalar) and τ the Euclidean angle of the osculating metric at $(x_0, y_0; 1, 0)$ then

$$I(x_0, y_0; 1, 0) = 2\lim_{\tau\to 0} (\phi - \tau)/\tau^2.$$

H. Busemann (Chicago, Ill.).

Milgram, Arthur N. On shortest paths through a set. Rep. Math. Colloquium (2) **2**, 39–44 (1940). [MF 5774]

Let M be any subset of a compact metric space S which is contained in at least one path of finite length. The author shows: (i) there is at least one shortest path in S containing M, and (ii) there is a maximal ("saturated") shortest path, that is, one not contained in any other. An interesting example shows that (ii) is not trivial. Further related results are given. *G. Birkhoff* (Cambridge, Mass.).

TOPOLOGY

***Hurewicz, Witold and Wallman, Henry. Dimension Theory.** Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941. vii+165 pp. $3.00.

The authors' aim is "to give a connected and simple account of the most essential parts of dimension theory," choosing those topics "which are of interest to the general worker in mathematics as well as the specialist in topology." In this they have been eminently successful. Since the appearance of Menger's Dimensionstheorie [B. G. Teubner, Leipzig, 1928], there have been two general accounts of somewhat restricted portions of dimension theory in the works on topology of Alexandroff and Hopf [Topologie, J. Springer, Berlin, 1935] and of Kuratowski [Topologie, Warsaw, 1933], each of which was a great improvement in simplicity and conciseness over the book of Menger. The work under review is a similar improvement, but covers the main portions of the theory; it goes also into various topics not appearing in earlier works.

Some topics omitted are: a study of weakly and strongly n-dimensional spaces, the complete form of the "Zerlegungssatz," and the theorem of Hurewicz on mappings which increase dimension. The theory of Alexandroff on dimension and homology is represented by one fundamental theorem [see below] in the final chapter; the present form and proof of the theorem is made possible by the use of coefficients mod 1 in the homology groups. In the course of the book various important topological theories are gone into, immediately useful in the sequel, such as the Brouwer fixed point theorem, the approximation to spaces by polytopes and properties of mappings into spheres. The last chapter gives a complete account of that part of combinatorial topology, with extension to compact metric spaces, which is needed for the final theorem. Because of its restricted nature, the chapter is not a good text for a beginning student to use, but it is an excellent example of an application of combinatorial methods to a geometrical problem. A main feature of the work is to give, in nearly each chapter, one or more new definitions of dimension, and to show its equivalence to the original definition.

An outline of the contents follows. I. Introduction. It is noted that all spaces will be separable metric. II. Dimension 0. Separation of sets, the sum theorem, etc. A theorem of Urysohn [Math. Ann. **94**, 262–295 (1925), in particular, p. 284] plays a fundamental role here; the reviewer would have liked to see it proved. III. Dimension n. Examples; rational points in Euclidean and Hilbert spaces. Theorems on sums and products of spaces. Decomposition theorems. If C_1 and C_2 are disjoint closed subsets of X, $A\subset X$, and dim $(A)\leqq n$, then C_1 and C_2 can be separated by a closed set B with dim $(AB)\leqq n-1$. Compact spaces. IV. Euclidean spaces E_n. The n-sphere S_n cannot be shrunk within itself to a point. The Brouwer fixed point theorem. Dim $(E_n)=n$. The Lebesgue covering theorem. The n-dimensional subsets of E_n are those containing open sets. Disconnecting E_n. V. Covering and imbedding theorems. The space Y^X of continuous mappings of X into Y plays a basic role. Coverings, of order n, of n-dimensional spaces. Such spaces may be put in E_{2n+1}; the extension of Hurewicz. A universal n-dimensional space. Approximation by polytopes. VI. Mappings in spheres. Dimensionality in terms of stability of mappings into E_n. Dim $(X)\leqq n$ if and only if a mapping of any closed subset into S_n is extendable over X. Various properties of homotopy. Mappings lowering dimension. Cantor manifolds. Invariance of domains in E_n. Separating sets in E_n. VII. Dimension and measure. Dim $(X)\leqq n$ if and only if X is homeomorphic to a bounded subset of E_{2n+1} whose $(n+1)$-dimensional measure is 0 (Szpilrajn). VIII. Homology and dimension. Homology and cohomology groups in a complex. Invariance under subdivision. Duality theory of Pontrjagin. Simplicial mappings; degree. Extension to compact spaces (using finite open coverings). Topological invariance. Duality. Hopf's extension theorem, first for polytopes, then for compact spaces. A compact X of finite dimension is of dimension not greater than n if and only if each cohomology class of any subset is part of a cohomology class of X, or, if and only if no nontrivial cycle in any subset bounds in X. Appendix. It is shown that fundamental properties of dimension do not hold in more general spaces.

The book is fairly free from errors; most of these are typographical. The equation on top of p. 22 must be misstated. The proof of Corollary 1 on p. 48 is false. In the final chapter, introduction of the Kronecker index (between cycles and cocycles) would make some of the proofs clearer. In general, the book provides uniformly interesting reading, and can be warmly recommended to anyone seriously interested in this side of topology. *H. Whitney.*

Löwig, Henry. Intrinsic topology and completion of Boolean rings. Ann. of Math. (2) **42**, 1138–1196 (1941). [MF 5819]

The author first shows how to introduce, by a new technical device, an intrinsic sequential topology into a Boolean

ring not assumed to be σ-complete. In the σ-complete case, this specializes to the usual lim sup-lim inf sequential topology. He then shows by an example that the intrinsic topology of a subring need not be that obtained by relativization (in the sense of Kuratowski) from the whole ring; when it is, he calls the subring "invariant." Next, he points out that every "invariant" extension of a Boolean ring A consists of joins of elements of A, and reproves MacNeille's theorem that A can be ("invariantly") completed by cuts. Seventeen inequalities on lim sup and lim inf are proved valid when the limits exist. The completion of A by fundamental sequences modulo null sequences is then discussed; this method of completion involves complications of the kind often confronted when sequences alone, and not directed sets, are used. The author shows how to surmount these difficulties within the theory of ordinary sequences and transfinite ordinals. Finally, the relation to closure-neighborhood topology is discussed. *G. Birkhoff.*

Milgram, A. N. Partially ordered sets and topology. Rep. Math. Colloquium (2) **2**, 3–9 (1940). [MF 5768]

An upper section of a partially ordered set P is defined as a subset U of P which contains with any element a all $b \geqq a$, and a separating system as a collection Σ of "upper sections" such that, given $a < b$, there exists $U \in \Sigma$ containing b but not a. Among other things, a characterization is given, in terms of these concepts, of those partially ordered sets which are isomorphic with bases of closed sets of a topological space. The concepts are also shown to permit an inductive formulation of the Heine-Borel covering theorem and the Brouwer reduction theorem. *G. Birkhoff.*

Frink, Orrin, Jr. Topology in lattices. Trans. Amer. Math. Soc. **51**, 569–582 (1942). [MF 6553]

The author examines various topologies definable in a lattice in terms of the order relation. The two topologies treated first are those of the Moore-Smith convergence of directed sets, and the "interval" topology in which the closed intervals $a \leqq x \leqq b$ (or $a \leqq x$, or $a \geqq x$) form a subbasis for the closed sets (that is, finite unions of such closed intervals form a "multiplicative" basis for the closed sets). These topologies coincide for a lattice which is the direct product of simply ordered lattices, but may be distinct otherwise. It is proved, and in a very neat way, that any complete lattice is a bicompact space in its interval topology; on the other hand, starting with the compact metric space made up of the points $1/n$ $(n=1, 2, \cdots)$, together with 0, topologized in the usual way, and then considering the complete lattice of all its closed sets, the author shows that a complete lattice need not be bicompact in its Moore-Smith topology. These two results answer unsolved problem 11 of Garrett Birkhoff's "Lattice Theory" [American Mathematical Society, New York, 1940]. It is further proved that the interval topology of an arbitrary lattice is weaker than or equivalent to the Moore-Smith topology (the usage of "weaker" being that a weaker topology assigns larger closures).

For the lattice of the closed sets of a topological space it turns out to be more satisfactory to modify the interval and Moore-Smith topologies and get new topologies called the "neighborhood" and "convergence" topologies. It is shown that for the lattice of closed sets of an arbitrary regular T_1-space S, the convergence topology is weaker than or equivalent to the neighborhood topology; if S is a bicompact Hausdorff space, the two lattice topologies are equivalent. The paper concludes with a discussion of continuous convergence. *H. Wallman* (Cambridge, Mass.).

Snyder, Walter S. A remark on the cardinal of limit spaces. Bull. Amer. Math. Soc. **48**, 121 (1942). [MF 6189]

The author claims to give two methods of introducing an L^* topology into a nonvacuous set S so that the resulting space is compact. In the first method S is normally ordered so that it has a last element. Then lim sup a_n is defined as the first element of the order which precedes only a finite number of the terms of (a_n), and lim inf a_n as the first element of the order which is preceded by, or equals, infinitely many terms of (a_n). The sequence is called convergent if these two elements coincide and the common value is its limit. These definitions do not determine an L^* topology. For, if a is the first element of the ordering of S and $a_n = a$ for all n, lim a_n is not defined and S is not an L^* space.

In the second method an element x_0 in S is fixed, lim $a_n = a$, by definition, if $a_n = a$ for $n > n_0$ and lim $a_n = x_0$, by definition, if no element of S or only x_0 occurs infinitely often among the a_n. *L. W. Cohen* (Lexington, Ky.).

Balanzat, Manuel. On the spaces D_0. Univ. Nac. Tucumán. Revista A. **2**, 169–175 (1941). (Spanish) [MF 6757]

The author modifies the postulates for a metric space, retaining only the following: (1) $xy \geqq 0$, (2) $xx = 0$, (3) $xy \leqq xz + yz$, and investigates the topological properties of the resulting space. Since symmetry of the metric is not required, three different types of accumulation element may be defined. Also, since $xy = 0$ does not necessarily imply $x = y$, the author introduces the sets of points $N_1(p)$, $N_2(p)$ and $N_3(p)$ which are, respectively, at zero distance on the left, on the right and on both left and right from p. Most of the paper is devoted to finding relations between these sets and the various types of limit element of a sequence. In addition, the author finds necessary and sufficient conditions on the neighborhoods of a topological space so that a metric satisfying (1), (2) and (3) and having the same topological properties may be defined. *J. V. Wehausen.*

Wallace, A. D. Reducible properties of Peano spaces. Anais Acad. Brasil. Sci. **14**, 47–49 (1942). [MF 6774]

The author considers a Peano space S and defines a closed set X in S to be an H-retract of S provided there exists an A-set reversing transformation [see G. T. Whyburn, Amer. J. Math. **58**, 305–312 (1936), in particular, p. 311] $T(S) = X$ which is the identity transformation on X. He shows that the collection of all H-retracts of S coincides exactly with the collection of A-sets in S. Furthermore, if $T(S) = X$ is an H-retraction and K is a subcontinuum of S which meets X, then $T(K) = X \cdot K$. The author calls a property P of continua reducible provided that if K is a subcontinuum of S having property P and if X is any A-set in S then the continuum $X \cdot K$ also has property P. It follows immediately from the theorem above that every property P which is invariant under monotone transformations is a reducible property. This fact, together with certain known monotone invariants, is used to give examples of several reducible properties of Peano spaces.

D. W. Hall (Providence, R. I.).

Harrold, O. G., Jr. A mapping characterization of Peano spaces. Bull. Amer. Math. Soc. **48**, 561–566 (1942). [MF 7049]

Let M denote a compact locally connected continuum, and K the subset of M consisting of all points which are interior points of free arcs of $M-a-b$. Denote the unit interval by I. A continuous mapping f of I into a subset of M is said to have property P provided that for any point x of K the set $f^{-1}(x)$ contains at most two points. The author proves that for any pair of points a and b of M there exists a continuous transformation $f(I)=M$ having property P and such that $f(0)=a$ and $f(1)=b$. This leads to the following interesting characterization of Peano spaces. Let a and b be points of the metric space X. Then in order that X be a Peano space it is necessary and sufficient that for any countable subset P of $Q-a-b$ there be a continuous mapping $f(I)=X$ such that $f(0)=a$, $f(1)=b$ and for any point y of P the set $f^{-1}(y)$ contains at most two points. The set Q is defined in the following way. Let K be defined as above and put $W=X-\bar{K}$. Then $Q=K+W\cdot N$, where N is the set of nonlocal separating points of X. *D. W. Hall.*

Ayres, W. L. A new proof of the cyclic connectivity theorem. Bull. Amer. Math. Soc. **48**, 627–630 (1942). [MF 7059]

This note gives an interesting proof of the cyclic connectivity theorem using only the following properties of Peano spaces: (a) every component of an open set is open; (b) open connected sets are arc-wise connected.

D. W. Hall (Providence, R. I.).

***Cohen, L. W. Uniformity in topological space.** Lectures in Topology, pp. 255–265. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

The author discusses a method of defining uniform properties for nonmetric spaces. Let S be a set of elements p and A a set of elements α. Suppose that to each p of S and α of A there corresponds a unique subset $U_\alpha(p)$ of S. The space whose points are the elements p and whose neighborhoods are the sets $U_\alpha(p)$ is denoted by S_A. The uniform properties in S_A are those which may be defined in terms of the $U_\alpha(p)$ for each α and all p. Examples are given such as (a) the property of being totally bounded, (b) the property of being a Cauchy sequence, (c) the property of being uniformly locally connected. A function $f(p)$ on S_A to $S_{A'}$ is uniformly continuous on a subset M of S_A if for each α' of A' there is an $\alpha=\alpha(\alpha')$ in A such that, for all p in M, $f[MU_\alpha(p)]\subset U_{\alpha'}[f(p)]$. The uniform homeomorphisms are the mappings which are 1-1 and uniformly continuous in both directions. From the point of view of mappings the uniform properties of a space are those which are invariant under uniform homeomorphisms.

The second section of the paper considers the case where A is denumerable, an example of such a space being a Hausdorff space satisfying the first denumerability axiom. Certain classical theorems holding in metric spaces are extended to such spaces. For example, if S_A is complete then every totally bounded set M is compact. The author then adds a symmetry axiom on S_A imposing the condition that if p and q be any two points of S_A and if q lies in a neighborhood of p then p lies in a neighborhood of q. Using this axiom he gets the results of Baire that if S_A is regular and complete then it must be of the second category, while, if $\{f_n(p)\}$ is a sequence of continuous functions on S_A to a metric space converging to a limit $f(p)$, then the set of points of continuity of $f(p)$ is dense in S_A.

The third section considers spaces S_A which are regular Hausdorff spaces and for which A is denumerable. This section is particularly concerned with continuous functions. It is shown that for each p of S_A there is a real function $f_p(q)$ on S_A which is lower semicontinuous, continuous and equal to zero at p, and bounded from zero in the complement of each neighborhood of p. The author uses such functions $f_p(q)$ to deduce conditions under which the space will be metrizable. The concluding section gives examples of certain known spaces which may be characterized as S_A spaces under suitable restrictions. *D. W. Hall.*

Albert, G. E. and Youngs, J. W. T. The structure of locally connected topological spaces. Trans. Amer. Math. Soc. **51**, 637–654 (1942). [MF 6556]

The authors consider a class of elements X topologized by the designation of certain subsets of X as open sets. The only restrictions on the topology are that (i) X is open and the null set is open, (ii) the sum of any number of open sets is open, (iii) the product of two open sets is open. A space X satisfying these conditions is called a topological space. The space X is said to be locally connected provided every component of an open set in X is an open set. Only locally connected topological spaces are considered in the paper. Two points a and b of X are said to be separated by a point p of X provided p is a closed set and a and b lie in different components of $X-p$. A point p is a cut point of X provided it separates two points a and b in X. The notion of conjugate points is then introduced in the classical manner [see Kuratowski and G. T. Whyburn, Fund. Math. **16**, 305–331 (1930)]. Using conjugacy as a relation the ideas of coherent and complete sets are introduced as in a paper by Radó and Reichelderfer [Duke Math. J. **6**, 474–485 (1940); these Rev. **1**, 318]. Any nondegenerate, complete and coherent set is called an N-set. It is shown that every N-set is closed, that the product of two distinct N-sets N_1 and N_2 is either vacuous or a single point x which separates any point of N_1-x from every point of N_2-x, and that there exists a unique N-set containing any pair of conjugate points. The kernel $k(N)$ of an N-set N consists of all points x in N such that x is not a frontier point of any component of $X-N$. A subset M of X which is an N-set with a nondegenerate kernel is called a true cyclic element of X provided it satisfies one of the characteristic properties of A-sets, that is, if S be any component of $X-M$ then $F(S)$ is a single point. Singular points of the space are defined as points which are not cut points or which lie in no true cyclic element. The degenerate cyclic elements of X consist of the cut points and the singular points of X. The most striking result proved is that the product of any connected set by a true cyclic element is vacuous or connected, which is, of course, another characteristic property of A-sets. Using this result it follows easily that every true cyclic element is a connected and locally connected topological space. If the space X be a locally connected continuum the same theory results using either the well-known cyclic element theory of G. T. Whyburn or this new approach. Among the results which the authors do not obtain, but which does hold for locally connected continua, is the fact that for every point x of X which is neither a cut point nor an end point of X there exists at least one point y of X conjugate to x.

By X_k the authors denote the set having as elements the cyclic elements (both true and degenerate) of X. It was shown in the second section of the paper that every point x

of X lies in a uniquely determined smallest cyclic element of X. They define a transformation $T(X)=X_h$ by associating with each point of X the smallest cyclic element of X which contains it. This transformation enables them to topologize X_h by calling a subset Z of X_h open if and only if $T^{-1}(Z)$ is an open set. It is easily seen that with this conception of open set X is a topological space. The authors devote the third section of their paper to showing that this hyperspace X_h is locally connected and acyclic.

A subclass H of the class B of all locally connected topological spaces is called hereditary provided that if X is in H then the hyperspace X_h is in H and each true cyclic element of X is in H. It is easily seen that the smallest hereditary class Y containing all Peano spaces consists exactly of all spaces X which are either Peano spaces or the hyperspaces of Peano spaces. The authors call a subclass Z of B invariant provided every strongly continuous image of a member of Z is itself a member of this class, and point out that every invariant subclass of B is hereditary. After mentioning several classes of spaces which are easily shown to be invariant (and hence hereditary) they prove that the hereditary property is enjoyed by each of the following classes of spaces: (a) all locally connected T_0-spaces, (b) the totality H_1 of all spaces in B for which any N-set is the closure of its kernel, (c) the class H_2 of all spaces X in H_1 having the property that, if x is any point, then the frontier of any component of $X-x$ is a single point, (d) the class H_3 consisting of all spaces X in H_2 such that if N is any N-set then the frontier of any component of $X-N$ is a single point.

Finally, denote by K the class consisting of all locally connected topological spaces which are (1) strongly continuous images of the closed unit interval, and (2) T_0-spaces in H_3. It is shown that K is an hereditary class containing Y and that every N-set in any space X of K is either a true cyclic element or a set which is the closure of a single point.

D. W. Hall (Providence, R. I.).

Freudenthal, Hans. Neuaufbau der Endentheorie. Ann. of Math. (2) **43**, 261–279 (1942). [MF 6462]

The "ends" of a space are "ideal points" whose adjunction serves to compactify the space. The compactification sought is by means of a maximal zero-dimensional endpoint set. In intuitive cases, this is the "natural" compactification; for example, it adds one ideal point to the plane, two to the unbounded cylinder. The paper sums up, strengthens and perhaps completes earlier results by the author principally, and by H. de Groot and the reviewer also. Let R be a topological space satisfying the second countability axiom. Let R be semicompact (each point possessing a neighborhood-basis with compact boundaries) and let the space of components of R be compact. Then (I) there exists a uniquely determined compact space R^* satisfying the second countability axiom such that R is topologically imbedded in R^* and R^*-R is zero-dimensional. Moreover, (II) if S denotes another compact space satisfying the second countability axiom such that R is topologically imbedded in S and $S-R$ is zero-dimensional, then there exists a continuous mapping $f(R^*)=S$ which is the identity on R. The paper closes with the following striking theorem (extending an older result of the author): a connected, semicompact topological group satisfying the second countability axiom is, of necessity, locally compact.

L. Zippin (Flushing, N. Y.).

Kelley, J. L. Hyperspaces of a continuum. Trans. Amer. Math. Soc. **52**, 22–36 (1942). [MF 6992]

The author considers a compact metric continuum X and makes an extensive study of the topological properties of the hyperspaces 2^X and $C(X)$ of X. Throughout the paper capital letters denote closed sets. By 2^X is meant the space of all closed and nonvacuous subsets of X metrized with the usual Hausdorff metric. The closed subspace of 2^X consisting of all subcontinua of X is denoted by $C(X)$. It is well known [see H. Whitney, Ann. of Math. (2) **34**, 244–270 (1933), in particular, p. 246] that there exists a real-valued function $\mu(A)$, continuous on 2^X, having the properties (i) if A is a proper subset of B then $\mu(A)<\mu(B)$; (ii) $\mu(X)=1$, $\mu(A)=0$ for any set A consisting of a single point. The author defines a segment from A_0 to A_1, where these sets are elements of 2^X, as a continuous mapping A_t of the unit interval into 2^X which satisfies the following conditions: (a) $\mu(A_t)=(1-t)\mu(A_0)+t\mu(A_1)$. He shows that, given $A_0, A_1 \varepsilon 2^X$, there will exist a segment from A_0 to A_1 if and only if $A_0 \subset A_1$ and every component of A_1 intersects A_0. Using segments an easy proof of the following theorem [which has been proved by Mazurkiewicz for 2^X] is obtained: Each of 2^X and $C(X)$ is the continuous image of the Cantor star, and hence arcwise connected.

For any element A of 2^X the set $\phi(A)$ is defined as the subset of 2^X consisting of all elements $\{a\}$ of 2^X, where $a \varepsilon A$. A space X is contractible in a space Y provided $X \subset Y$ and the identity transformation on X is homotopic to a constant in Y. The author shows that the following properties are equivalent: (a) $\phi(X)$ is contractible in 2^X; (b) 2^X is contractible; (c) $C(X)$ is contractible. He obtains another condition under which 2^X and $C(X)$ will be contractible and which generalizes a theorem of Wojdyslawski [see Fund. Math. **30**, 247–252 (1938)].

The author proves that the space 2^X is acyclic in all dimensions and leaves open the question as to whether or not the same theorem is true for $C(X)$. He then gives an easy proof of the theorem of Wojdyslawski to the effect that the following properties are equivalent: (a) X is Peanian; (b) 2^X is Peanian; (c) $C(X)$ is Peanian; (d) 2^X is an absolute retract; (e) $C(X)$ is an absolute retract. The following extension theorem is obtained: Let Y be a compact locally connected subset of a metric space Z and let $f(Y)$ be a continuous mapping of Y into 2^X (or $C(X)$). Then f can be extended to a continuous mapping of Z into 2^X (or $C(X)$). Using the concept of dimension it is shown that 2^X always contains the homeomorph of the fundamental cube and that if X is Peanian then $C(X)$ is finite dimensional if and only if X is a linear graph.

Next local separating points of Peano spaces are investigated and it is shown: (i) If $f(E^2)=E^1$ is a continuous mapping of the unit square onto the unit interval then there exists a closed totally disconnected subset Z of E^2 such that $f(Z)=E^1$. (ii) Let $f(X)=Y$ be a monotone interior transformation of a compact metric space X into a set Y of finite dimension. Then there exists a closed totally disconnected subset X_0 of X mapping onto Y under f if and only if the set of points on which f is 1-1 is a totally disconnected subset of Y.

Finally it is shown: (i) In order that a continuum X be indecomposable it is necessary and sufficient that $C(X)-X$ fail to be arcwise connected. (ii) The continuum X is a Knaster continuum if and only if $C(X)$ contains a unique arc between every pair of its elements. (iii) If X is a Knaster continuum, then for every $\epsilon>0$ there exists a monotone in-

terior transformation $f(X)=Y$ such that $0<\text{diam } f^{-1}(y)<\epsilon$ for all y of Y. (iv) If X is a Knaster continuum, if the transformation of (iii) exists, and if Y is of finite dimension, then $\dim X=1$. (v) If there exist Knaster continua of dimension greater than one then there exist Knaster continua of infinite dimension. *D. W. Hall.*

Gama, Lelio I. Sur quelques points de la théorie des espaces abstraits et la notion d'accumulatif. Anais Acad. Brasil. Sci. **12**, 69–83 (1940). [MF 6572]

The author proves theorems concerning (1) hereditary properties of sets, (2) the norm of a set of sets, (3) accumulation set of a family of sets, (4) limit set of a family of monotone sets, etc., the most general basic space being a (V) space. The derivation of the Borel-Lebesgue covering theorem from a lemma to the effect that in a metric space every hereditary property of a compact set K is localizable at a point of the closure of K [the definitions of "hereditary" and "localizable" differing from the usual ones [e.g., Kuratowski, Topologie, Warsaw, 1933, v. I, p. 29]] is noteworthy, though the proof offered for the lemma seems to the reviewer incomplete. It appears that the author's use of the notion of hereditary property is not entirely in agreement with his definition of the concept.

L. M. Blumenthal (Columbia, Mo.).

Otchan, G. Sur un théorème de Baire. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 427–430 (1941). (Russian. French summary) [MF 6832]

The following theorem is proved: In a compact, completely regular (that is, each point is a G_δ) space the sum (intersection) of a well-ordered, nondecreasing (nonincreasing) system of closed (open) sets is either a closed (open) set or an $F_\sigma(G_\delta)$. For a space with a denumerable basis this is a consequence of the following theorem of Baire: Every well-ordered, monotonic system of open (or closed) sets contains at most denumerably many different sets.

J. V. Wehausen (Columbia, Mo.).

Fox, Ralph H. A characterization of absolute neighborhood retracts. Bull. Amer. Math. Soc. **48**, 271–275 (1942). [MF 6405]

The following theorem is proved: For a separable metrizable space X the following conditions are equivalent: (1) X is an absolute neighborhood retract; (2) there is a homeomorphism f of X into Q (the Hilbert parallelotope), such that $f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0,1]$; (3) $f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0,1]$ for every homeomorphism f of X into Q. A similar theorem holds for absolute retracts, obtainable from the above theorem by deleting "neighborhood" from (1), (2) and (3). [On p. 274, line 10, read "V' of $X\times[0]$ in $Q\times[0,1]$" instead of "V' of $X\times[0]+Q\times(0,1]$."] *H. M. Gehman* (Buffalo, N. Y.).

Clark, C. E. On 3-dimensional manifolds. Bull. Amer. Math. Soc. **48**, 437–439 (1942). [MF 6710]

The author considers a 3-dimensional manifold P and a 2-dimensional manifold Q imbedded in P. The terminology throughout the paper is that of Seifert-Threlfall [Lehrbuch der Topologie, Teubner, Leipzig, 1934]. All manifolds considered are finite while the simplexes and cells are closed point sets. It is assumed that P and Q admit of a permissible simplicial division K, that is, a simplicial division of P such that some subcomplex, say L, is a simplicial division of Q. The ith normal subdivisions of K and L are denoted by K_i and L_i, respectively. The neighborhood N_i of L_i is the simplicial complex consisting of the simplexes of K_i which have at least one vertex in L_i together with the sides of all such simplexes. The boundary B_i of N_i is defined as the simplicial complex consisting of the simplexes of N_i that have no vertex in L_i. The author proves that the boundary B_2 is a two-fold but not necessarily connected covering of Q, and any change of the permissible division K replaces B_2 by a homeomorph of itself. He gives an example to show that the theorem does not hold if B_2 is replaced by B_1. He states that he can prove the theorem by the same method but with much less effort if P and Q are replaced by 2- and 1-dimensional manifolds. *D. W. Hall.*

Hopf, Heinz. Fundamentalgruppe und zweite Bettische Gruppe. Comment. Math. Helv. **14**, 257–309 (1942).

This paper is, in the reviewer's mind, one of the most important contributions to combinatorial topology in recent years. It gives far reaching results concerning the relations between the fundamental group, the first and second homology and cohomology groups, and the products between these groups, with beautiful and simple methods. The work is based on some new constructions in groups which are undoubtedly of real significance by themselves. The paper is in three main parts: the group theory; determination of the second homology group $\mathfrak{B}^2$ (all groups are with integer coefficients) modulo the group $\mathfrak{S}$ of "spherical homology classes" [see below]; and the determination of the products for dimensions not greater than 2 (omitting considerations of torsion). In each case, the formulas are in terms of the fundamental group $\mathfrak{G}$, and pure group-theoretic constructions, but with geometric meanings.

Let $\mathfrak{G}$ be a group with finite sets of generators and relations. Let Γ be the set of all $\alpha=(X_1, Y_1, \cdots, X_n, Y_n)$, $X_n, Y_n\in\mathfrak{G}$. Two elements are added by placing them in juxtaposition, and α^{-1} is α with the X_i and Y_i interchanged. Let Δ be the subset of Γ with elements α with the following properties: For each X_i and Y_i fixed representations as products of elements of $\mathfrak{G}$ may be chosen so that $C(\alpha)=X_1Y_1X_1^{-1}Y_1^{-1}\cdots Y_n^{-1}$, when written out, is also of the form $U_1V_1U_1^{-1}V_1^{-1}\cdots V_m^{-1}$, with fixed representations of the U_i and V_i, and with each $V_i=1$. (Hence $C(\alpha)=1$.) Then Γ falls into cosets modulo Δ; these form a group $\mathfrak{G}^*$. If F is a homomorphism of a free group $\mathfrak{F}$ onto $\mathfrak{G}$, with kernel $\mathfrak{R}$, and $\mathfrak{C}(\mathfrak{R})$ is the subgroup generated by all $frf^{-1}r^{-1}$ ($f\in\mathfrak{F}$, $r\in\mathfrak{R}$), then Δ consists of all α such that a_i and b_i exist with $F(a_i)=X_i$, $F(b_i)=Y_i$, and $C(a_1\cdots b_n)\in\mathfrak{C}(\mathfrak{R})$. Hence, if $\mathfrak{C}=\mathfrak{C}(\mathfrak{F})$, $\mathfrak{G}^*\cong\mathfrak{C}/\mathfrak{C}(\mathfrak{R})$. Let $\mathfrak{G}_1^*$ be the subgroup of $\mathfrak{G}^*$, using only those $\alpha\in\Gamma$ with $C(\alpha)=1$. Then $\mathfrak{G}_1^*\cong(\mathfrak{C}\cap\mathfrak{R})/\mathfrak{C}(\mathfrak{R})$. Examples: If $\mathfrak{G}$ is free, $\mathfrak{G}_1^*=0$. If $\mathfrak{G}$ is a free Abelian group with p generators, $\mathfrak{G}_1^*$ is a free Abelian group with $\frac{1}{2}p(p-1)$ generators.

It is easily seen that any 2-chain Y in the complex K may be determined as the image of an oriented 2-cell E; the boundary curve ρ of E is mapped into a closed path r, called a "homotopy boundary" (h. b.) of Y. Let $\mathfrak{G}$ and $\mathfrak{F}$ be the fundamental groups of K and K^1 (=1-dimensional part of K); let F be the natural homomorphism of $\mathfrak{F}$ onto $\mathfrak{G}$, with the kernel $\mathfrak{R}$. Then r is a h. b. if and only if it determines an element of $\mathfrak{R}$. The different h. b.'s of Y differ by elements of $\mathfrak{C}(\mathfrak{R})$; Y is a cycle if and only if $r\in\mathfrak{C}$. If Y is a "spherical cycle," obtainable as the image of a 2-sphere, this is equivalent to mapping the above ρ into a point; thus $r\in\mathfrak{C}(\mathfrak{R})$. The mappings are homomorphic; it follows that

$$\mathfrak{B}^2/\mathfrak{S}^2\cong(\mathfrak{C}\cap\mathfrak{R})/\mathfrak{C}(\mathfrak{R})\cong\mathfrak{G}_1^*.$$

As a corollary, if $\mathfrak{G}_1^* \neq 0$, $\mathfrak{B}^2 \neq 0$. The structure of $\mathfrak{G}_1^*$ is determined in various examples, with the help of specific complexes with $\mathfrak{S}^2 = 0$.

The third part of the paper is carried out for intersections in a manifold, but holds for products in a complex. The basic formula is the following: Let U be a 1-cocycle, and Z a 2-cycle. If r is a h. b. of Z, then $r \in \mathfrak{C}$, and corresponds to an element of $\mathfrak{G}$ of the form $C(X_1, \cdots, Y_h)$. Each X_i and Y_i defines 1-cycles X_i' and Y_i'. Then

$$U \cap Z \sim \sum_{i=1}^{h} [(U \cdot Y_i') \cdot X_i' - (U \cdot X_i') \cdot Y_i'].$$

A similar formula holds for the $\cup$ product. The proof runs as follows. Since r is a h. b., the path $X_1 Y_1 X_1^{-1} Y_1^{-1} \cdots Y_h^{-1}$ is the image of the boundary ρ of a simplex E under a mapping f. If we identify the parts of E going into X_1 and X_1^{-1}, etc., a closed oriented surface ξ^2 is formed; $Z \sim f(\xi^2)$. Since $U \cap f(\xi^2) \sim f(f' U \cap \xi^2)$ (f' = dual of f), it is sufficient to prove the formula for surfaces ξ^2; in ξ^2 it reduces to well-known facts.

When the chains are expressed as above in terms of $\mathfrak{G}$ (the products act trivially on elements of $\mathfrak{S}$), a product is defined, as follows. Let $\mathfrak{G}' = \mathfrak{G}/\mathfrak{C}$ (= first reduced homology group), with character group (into the integers) Ch $\mathfrak{G}'$ (= first reduced cohomology group). Then the product is between Ch $\mathfrak{G}'$ and $\mathfrak{G}_1^*$, into $\mathfrak{G}'$, or between Ch $\mathfrak{G}'$ and Ch $\mathfrak{G}'$ into $\mathfrak{G}_1^*$. The product may be defined algebraically, with the help of results of E. Witt [J. Reine Angew. Math. 177, 152–160 (1937)]. Various corollaries and examples are given, in particular, for 3-dimensional manifolds.

H. Whitney (Cambridge, Mass.).

Gysin, Werner. Zur Homologietheorie der Abbildungen und Faserungen von Mannigfaltigkeiten. Comment. Math. Helv. 14, 61–122 (1942).

This paper contains an investigation of fiber spaces in a direction different from that of Whitney and that of Hurewicz-Steenrod; it is concerned with homology properties of fiber spaces. Chapter I gives a short survey of the theory of products of cocycles in complexes [Whitney] and its connection with the intersection of cycles in manifolds, and discusses the properties of simplicial mappings: a simplicial mapping f of a complex K into a complex K' induces, as is well known, a mapping f' of the cochains of K' into those of K (the dual mapping). If K and K' are manifolds of dimension n and ν, respectively, then by going over to dual cells one obtains from f' a mapping φ of the p-chains of K' into the $(p+n-\nu)$-chains of K, which preserves the boundary [this is Hopf's inverse homomorphism]; f and φ are related by a well-known functional equation.

Chapter II contains the basic notion of the paper, that of the "H process," which applies to an arbitrary mapping of a manifold into a manifold of lower dimension: Let L and Λ be manifolds of dimension n and ν, and let f be a simplicial mapping of L into Λ. Let ζ be a cycle of Λ, the φ-image of which bounds in L; let C be a chain bounded by $\varphi(\zeta)$; put $f(C) = h(\zeta)$. Now let $d = n - \nu$ be greater than 0; then $h(\zeta)$ is a cycle, because the boundary of C is mapped by f in a degenerated manner. Let $\mathfrak{B}^p$ be the p-dimensional homology group of Λ; let $\mathfrak{K}^p$ be the subgroup of $\mathfrak{B}^p$ consisting of those ζ for which $\varphi(\zeta) \sim 0$; let $\mathfrak{F}^p$ be the subgroup of $\mathfrak{B}^p$ formed by those homology classes which contain f-images of cycles of L. Then $h(\zeta)$ induces a homomorphic mapping H of $\mathfrak{K}^p$ into the rest class group $\mathfrak{B}^{p+d+1}/\mathfrak{F}^{p+d+1}$. It is proved that the mapping H depends only on the homotopy class of f. Hence, if H is not the 0-homomorphism, then f is not contractible. The multiplicative properties of H are studied.

In chapter III fiber spaces with a (homology-) sphere as fiber are considered: let f be such that the inverse image of each point of Λ is homeomorphic with a sphere S^d, and that the inverse image of a simplex ξ of Λ is equivalent to the topological product $\xi \times S^d$ ("local product definition"). By direct geometric reasoning it is proved that in this case the mapping H is an isomorphism of $\mathfrak{K}^p$ onto $\mathfrak{B}^{p+d+1}/\mathfrak{F}^{p+d+1}$. This has numerous applications; the mapping f is essential; the Betti numbers of L are not greater than the Betti numbers $\Lambda \times S$; if L is a sphere, then the homology ring of Λ is completely determined by n and d, and $d+1$ is a divisor of $n+1$; if d is even, then S is not ~ 0 in L; some results on the space of line elements of a manifold, etc.

H. Samelson (Princeton, N. J.).

Eckmann, Beno. Zur Homotopietheorie gefaserter Räume. Comment. Math. Helv. 14, 141–192 (1942).

This is a conscientious and detailed investigation of the homotopy properties of fibre-spaces; some of the main results, however, had been independently anticipated by Hurewicz-Steenrod [Proc. Nat. Acad. Sci. U. S. A. 27, 60–64 (1941); these Rev. 2, 323; further quoted as HS] and by Ehresmann-Feldbau [C. R. Acad. Sci. Paris 212, 945–948 (1941); these Rev. 3, 58; further quoted as EF]. Fibre-spaces are defined practically as in HS, that is, by the existence of a retraction, depending continuously upon the fibre, of the neighborhood of each fibre into the fibre (the author observes that part of the results subsist if, instead of such a "retractible" decomposition into fibres, one considers a "retractible" covering by closed sets). This definition does not imply that all fibres are homeomorphic; it does imply that they (or more generally the subsets of a "retractible" covering) all have the same homotopy type and the same dimension.

The basic tool is a lemma, identical with the "covering homotopy theorem" of HS (= "deformation lemma" of EF). Among other useful immediate consequences, it is shown that, if the identical mapping of the fibre-space E on itself is essential, so is its mapping ("projection") on the base-space (or decomposition-space) B; this generalizes a well-known theorem of Hurewicz on group-spaces. Formulae for the homotopy groups of fibre-spaces are then deduced, which, except for the greater generality of the assumptions, are identical with those of EF [see these Rev. 3, 58; as they generalize earlier results of Hurewicz on group-spaces, the author calls them "Hurewicz's formulae"]; the lemma, and a result identical with theorem 2 of HS, constitute the main steps in the proof. It is further shown that, if the fibre F in the fibre-space E (with decomposition-space B) is homotopic to 0 in E, the homotopy group $\pi_n(B)$ is the direct sum of $\pi_n(E)$ and $\pi_{n-1}(F)$ for $n \geqq 2$. On the other hand, it is shown that, if there exists a "surface of section" (that is, a map of B into E, the projection of which on B is the identical mapping of B into itself), $\pi_n(E)$ is the direct sum of $\pi_n(F)$ and $\pi_n(B)$, that is, the same as for a direct product $F \times B$.

The rest of the paper consists of a large number of special results on spheres and fibre-spaces connected with spheres, not all of which bear a direct relationship with the above-mentioned general theorems; it would have been more helpful if these had been clearly classified. Some of those results duplicate results in HS or earlier ones; among others, we

quote the following: (a) for every fibre-space E, where the base-space B is a sphere S^m and the fibre is a sphere S^{m-1}, a numerical invariant k is defined, which is 0 if and only if there is a "surface of section"; this has the value 2 for the space L_m of tangent unit-vectors to S^m, if m is even, and 0 if m is odd; (b) the homotopy groups $\pi_{m-1}(L_m)$, $\pi_m(L_m)$, $\pi_{m+1}(L_m)$ are of order 2 if m is even and not less than 4; (c) if two independent vector-fields can be found over the sphere S^m, then there exists a vector-field over S^m which is independent of an arbitrarily given vector-field over the same; (d) hence, and from a result of Pontrjagin, it is deduced that no two independent vector-fields can exist over S^3 (nor, a fortiori, can there be an absolute parallelism over S^3). *A. Weil* (Bethlehem, Pa.).

Eckmann, Beno. Über die Homotopiegruppen von Gruppenräumen. Comment. Math. Helv. **14**, 234–256 (1942).

If $f(x)$, $g(x)$ are mappings of a space S into a group-space G, then $f(x)\cdot g(x)$ (where $\cdot$ is the group-operation) is again such a mapping. This, applied to mappings of a sphere S^n into G which define elements of the homotopy-group $\pi_n(G)$, gives an operation between elements of $\pi_n(G)$; it is shown in § 1 that this is the same as Hurewicz addition. This is given as a particular case of a similar result for a space G where a (not necessarily associative) operation $\cdot$ has been given, and some further results, equally elementary, are proved for that case. In § 2, with the help of Freudenthal's "Einhängung," some cases are given where an application of degree k of the sphere S^m into itself induces the application $(\alpha\to k\cdot\alpha)$ of $\pi_n(S^m)$ into itself. In § 3, the homotopy-groups π_3, π_4, π_5 are determined for the orthogonal group in any number of variables; the method is that of fibre-spaces as developed in a previous paper of the same author [see the preceding review]. [N.B. Some of the results on spheres and the orthogonal groups in the last-mentioned paper, and the determination of π_3, π_4, π_5 for the orthogonal groups in this one, are identical with results independently obtained, by similar methods, by G. W. Whitehead, Ann. of Math. (2) **43**, 132–146 (1942); these Rev. **3**, 197.]
A. Weil (Bethlehem, Pa.).

Mayer, W. A new homology theory. I, II. Ann. of Math. (2) **43**, 370–380, 594–605 (1942). [MF 6467, 7014]

The new theory is based on considering nonoriented simplices (possibly with repeated vertices), taking their formal boundary and reducing all the chains modulo a fixed prime p. The boundary operator F thus obtained satisfies $F^p=0$ instead of the usual $F^2=0$. An n-chain K^n is called a (q, n)-cycle if $F^q(K^n)$ vanishes (mod p); dividing the group of the (q, n)-cycles by the subgroup of all cycles $F^{p-q}(K^{n+p-q})$, the (q, n)-homology group H_q^n is obtained. Following the Čech method the groups H_q^n are then defined for topological spaces. After the invariance of the groups of a complex under subdivision has been established, the following invariance theorem is proved: For geometric complexes the groups of the complex and those of the underlying space are isomorphic.

The second part of the paper deals with the appropriate generalization of the Euler-Poincaré formula and also gives some relations for the ranks of the new homology groups.
S. Eilenberg (Ann Arbor, Mich.).

Milgram, Arthur N. On iterated mappings. Rep. Math. Colloquium (2) **2**, 21–24 (1940). [MF 5771]

Single-valued transformations of an (infinite) class T onto powers T^n of itself are considered. It is shown how in general, from a map of T onto T^2, maps of T onto any T^n can be constructed; moreover, if the original map is continuous, so is the constructed map. This permits one to map the unit segment continuously onto the Hilbert cube; one construction in this case was already found by Sierpinski [Wiadom. Mat. **42**, 1–3 (1936)]. *G. Birkhoff.*

Kline, J. R. What is the Jordan curve theorem? Amer. Math. Monthly **49**, 281–286 (1942). [MF 6689]

Kneebone, G. T. The three houses problem. Math. Gaz. **25**, 78–81 (1941). [MF 5729]

MATHEMATICAL PHYSICS

Flügge, S. Die Eigenschwingungen eines Flüssigkeitstropfens und ihre Anwendung auf die Kernphysik. Ann. Physik (5) **39**, 373–387 (1941). [MF 6850]

The paper consists of two parts. The first part deals with the deformation oscillations of an incompressible, non-viscous liquid drop with no angular momentum and held together by surface tension. Under these circumstances each element of the liquid drop can be characterized by a velocity vector v which can be derived from a potential Φ. This velocity-potential Φ satisfies Laplace's equation $\Delta\Phi=0$. The solution for Φ is then sought in the standard form as a series in spherical harmonics:

$$\Phi=\sum\beta_{l,m}(t)r^lP_l^m(\cos\vartheta)e^{im\varphi}. \tag{1}$$

Similarly, if the surface of the liquid drop is represented by an equation of the form

$$R=R_0+\sum\alpha_{l,m}P_l^m(\cos\vartheta)e^{im\varphi}, \tag{2}$$

then there exist the relations (for small oscillations)

$$\dot\alpha_{l,m}=l\beta_{l,m}R_0^{l-1}. \tag{3}$$

The energy E of the liquid drop is then evaluated according to the equation

$$E=\tfrac{1}{2}\rho\int\int\int|\operatorname{grad}\Phi|^2r^2\sin\vartheta\,d\vartheta\,d\varphi\,dr + \tfrac{1}{2}\gamma\int\int\left\{\frac{1}{\sin^2\vartheta}\left[\frac{\partial}{\partial\varphi}(R-R_0)\right]^2 + \left[\frac{\partial}{\partial\vartheta}(R-R_0)\right]^2-2(R-R_0)^2\right\}\sin\vartheta\,d\vartheta\,d\varphi, \tag{4}$$

where ρ is the density and γ the coefficient of surface tension. In equation (4) the first term is a volume integral representing the kinetic energy of motion while the second term is a surface integral representing the potential energy due to surface tension. From equations (1), (2), (3) and (4) the author derives that

$$E=\sum_l\sum_m\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}\gamma(l-1)(l+2)\times\left\{|\alpha_{l,m}|^2+\frac{1}{\omega_l^2}|\dot\alpha_{l,m}|^2\right\}, \tag{5}$$

where

$$\omega_l=\left[\frac{\gamma}{\rho R_0^3}\,l(l-1)(l+2)\right]^{\frac{1}{2}}. \tag{6}$$

Accordingly, the ω_l's are the characteristic frequencies of the deformation oscillations.

The second part of the paper deals with the average rate of radiation of electromagnetic energy by a uniformly charged liquid drop when undergoing deformation oscillations of the type considered. The calculation consists in evaluating an appropriate integral of the Poynting vector. Applications of the formulae are made to nuclear physics on the Bohr model of the nucleus as a liquid drop. The most important conclusion reached appears to be the suggestion that, contrary to current views, nuclear isomers may correspond to the different states of deformation oscillations of a liquid drop with no angular momentum.

S. Chandrasekhar (Williams Bay, Wis.).

Schönberg, Mario. First order relativistic equations of motion in quantum mechanics. Anais Acad. Brasil. Sci. 11, 341–345 (1939). (Portuguese) [MF 6568]

It is pointed out that there are linear, relativistic, first order equations of motion for a free elementary particle other than those of Dirac. For instance,

$$p_{\lambda\dot\omega}a^{\dot\omega\dot\rho\cdots}_{\alpha\beta\cdots}=m_1cb^{\dot\rho\dot\rho\cdots}_{\dot\omega\lambda\alpha\beta\cdots},$$
$$p^{\dot\omega\lambda}b^{\dot\rho\dot\rho\cdots}_{\dot\omega\lambda\alpha\beta\cdots}=m_2ca^{\dot\rho\dot\rho\cdots}_{\alpha\beta\cdots},$$

where $a^{\dot\rho\dot\rho\cdots}_{\alpha\beta\cdots}$ and $b^{\dot\rho\dot\rho\cdots}_{\dot\omega\lambda\alpha\beta\cdots}$ are spinors symmetric in both the dotted and undotted indices, $m=(m_1m_2)^{\frac{1}{2}}$ is the mass of the particle and $p^{\lambda\dot\alpha}$ is the momentum-energy spinor.

A. Schwartz (State College, Pa.).

Schrödinger, Erwin. Exchange and spin. With a note by James Hamilton. Proc. Roy. Irish Acad. Sect. A. **47**, 39–52 (1941). [MF 6450]

Dirac's identity connecting the operators of exchange and spin which was given in the 1935 edition of his "Principles of Quantum Mechanics" for the case in which the spin number s has the value $\frac{1}{2}$ is established now for an arbitrary value of s, and a new proof is given of Dirac's relation in which use is made of the square of the resultant spin

$$1+\tau_{12}=M=(\sigma_{x1}+\sigma_{x2})^2+(\sigma_{y1}+\sigma_{y2})^2+(\sigma_{z1}+\sigma_{z2})^2.$$

The theorem is that, if ζ_1 and ζ_2 are the two spin variables, $\psi(\zeta_2,\zeta_1,\cdots)=\tau_{12}\psi(\zeta_1,\zeta_2,\cdots)$. The polynomial τ_{12} is next determined and it is shown that, if $M=n(n+1)$, $\tau_{12}=(-)^{2s}F(n+1,-n;1;1)$, where n is an operational variable. An appendix on double-spin algebra is added by James Hamilton, in which use is made of concise symbols for the components of the resultant spin. An antisymmetric triplet, that is, a sort of cross product of the spins, is defined and a proof is given of the alternating symmetry character of successive lines of Schrödinger's triangular scheme for the eigenfunctions of M and $\sigma_{z1}+\sigma_{z2}$. *H. Bateman.*

Tamm, Ig. Motion of mesons in electromagnetic fields. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 551–555 (1940). [MF 6267]

The equations for vector mesons (spin 1) in an external electromagnetic field are written in a form in which momentum and spin operators acting on a many component wave function appear explicitly (particle description). Stationary states with definite charge exist in spite of the inherent possibility of production and annihilation. The equations are set up for a Coulomb field. They show, owing to the intrinsic dipole moment of the mesons, higher singularities at the origin than Dirac's equations. In consequence the majority of the solutions are irregular at the origin so that the bounded ones do not form a complete set. Calculations of meson scattering at high energies by Coulomb fields are therefore open to grave doubts. *L. W. Nordheim.*

Ginsburg, W. L. Zur Theorie der Wechselwirkung des Mesons mit dem elektromagnetischen Feld. Acad. Sci. USSR. J. Phys. **5**, 47–57 (1941). [MF 5739]

The properties of vector mesons (spin 1) are investigated with the help of a particle equation of the Dirac type as introduced by Tamm [cf. the preceding review]. The usual perturbation mechanism is modified to take into account the absence of a positive definite charge density, and the known matrix elements for radiative transitions are derived. The self energy of the meson is calculated in the classical limit (neglecting the reaction in the emission of photons) and in this approximation no anomalous behavior is found. Mesons differ from electrons in that their magnetic moment contributes to radiation at relativistic energies. In this respect the mesons resemble more closely the classical dipoles than the Dirac electrons do. Finally the scattering of photons by a classical magnetic dipole is investigated. It is shown that neglecting the proper field of the dipole would lead to a cross section proportional to the square of the frequency, while the consideration of this field (causing an inertia perpendicular to the dipole axis) brings it down to a constant value. *L. W. Nordheim.*

Power, S. C. On the stability of crystal lattices. VII. Long-wave and short-wave stability for the face-centred cubic lattice. Proc. Cambridge Philos. Soc. **38**, 61–66 (1942). [MF 6035]

It is shown that a face centered cubic lattice, in which the forces fall off so rapidly with distance that only those between first neighbors must be considered, is stable against short wave displacements if it is stable against long wave deformations. *L. W. Nordheim* (Durham, N. C.).

Peng, H. W. and Power, S. C. On the stability of crystal lattices. VIII. Stability of rhombohedral Bravais lattices. Proc. Cambridge Philos. Soc. **38**, 67–81 (1942). [MF 6036]

The stability of the Bravais lattices with rhombohedral unit cell of arbitrary angle is investigated under the assumption that the potential contains two terms, each proportional to a reciprocal power of distance. It is shown that among the cubic Bravais lattices contained in this group the face and body centered ones correspond to a minimum of potential energy, but the simple cubic lattice to a maximum. By numerical calculation of the energy of intermediate lattices it is shown that no other extrema for the potential energy exist and that the face centered lattice corresponds to the absolute minimum. Finally, equilibrium conditions for compound lattices with a number of parameters are formulated and it is shown, under assumption of the above form of potential energy, that these conditions can be divided into one set for change of volume and an independent set for change of shape. *L. W. Nordheim.*

Born, M. On the stability of crystal lattices. IX. Covariant theory of lattice deformations and the stability of some hexagonal lattices. Proc. Cambridge Philos. Soc. **38**, 82–99 (1942). [MF 6037]

The theory of lattice deformations is presented in a new form using the tensor calculus. Two sets of coordinate sys-

tems are introduced, a space-fixed orthogonal system S and an oblique body-fixed system Σ, whose axes are parallel to the sides of a unit cell. All physical quantities are represented in such a way that their double covariance, against rotations of S and against different choice of the unit cell for Σ, is immediately evident. The lattice is described by a metric tensor defining Σ and a set of parameters giving the position of atoms within the unit cell. The expressions for the change in potential energy for periodic deformations are given in the double covariant form. The case of central forces is worked out in detail. With the new method the criteria for stability formulated in an earlier paper [M. Born, Proc. Cambridge Philos. Soc. 36, 160–172 (1940); these Rev. 1, 277] can be extended to cover more complicated cases. As an application it is shown, under assumption of central forces, that the Bravais hexagonal lattice is unstable but that the close-packed hexagonal lattice is stable. The elastic constants for the latter are also calculated.

L. W. Nordheim (Durham, N. C.).

Fürth, R. On the theory of finite deformations of elastic crystals. Proc. Roy. Soc. London. Ser. A. 180, 285–304 (1942). [MF 6931]

The theory of classic behavior is worked out for cubic crystals under retention of terms of third order in the potential energy. The stress-strain relations are obtained and the (five) elastic constants are expressed in terms of lattice sums. The latter are computed for the three types of cubic crystals (simple, face centered, body centered) under assumption of power laws for the attractive and repulsive forces between atoms with various sets of exponents. A number of second order effects (deviations from Hooke's law) are worked out in detail. They could be observed by suitable experimental arrangements and would provide data on the exponents in the force law. The only available check, at present, is given by Bridgman's measurements on compressibility at high pressures which lead to very reasonable values for the exponents. *L. W. Nordheim.*

Gogate, D. V. and Kathavate, Y. V. Effusion phenomena in a degenerate Bose-Einstein gas. Philos. Mag. (7) 33, 310–314 (1942). [MF 6594]

In this paper the effusion phenomena in a degenerate Bose-Einstein gas are investigated, taking into account the effect of relativity mechanics. It is shown that, in degenerate Bose-Einstein statistics, the expressions for the effusion of number, mass and energy depend only on temperature whereas, in the nondegenerate case, they depend upon concentration as well. The effusion from one chamber into another, the two being at different temperatures but containing the same gas, is then calculated and a simple and direct proof of the general theorem that "the net transport from chamber (1) to chamber (2) is equal to the difference between the transports from chamber (1) to vacuum and chamber (2) to vacuum" is also given.

B. O. Koopman (New York, N. Y.).

Rucker, J. T. An application of vector analysis to thermodynamics. Amer. Math. Monthly 49, 238–242 (1942). [MF 6511]

Vectorial interpretation and some extensions of the results of Gibbs on the geometric representation of the equations of thermodynamic equilibrium [J. W. Gibbs, Scientific Papers, New York, 1906, v. I, pp. 42, 115–129 or Connecticut Acad. Trans. 2, 382–404 (1873); 3, 108–248 (1876), 343–524 (1878)]. *I. Opatowski* (Chicago, Ill.).

De Donder, Th. À toute mécanique ondulatoire correspond une mécanique statistique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 438–440 (1941). [MF 6896]

In the well-known notation of wave mechanics, the average value of F is defined by $\bar{F}=\int\psi F\bar{\psi}$. The author shows how to change this integral into an integral taken in the phase-space. In this way an analogy with statistical mechanics is established. With every function ψ representing a state in a given system in wave mechanics there is associated a certain function of distribution, just as in statistical mechanics. *L. Infeld* (Toronto, Ont.).

Géhéniau, J. et Prigogine, I. Sur la mécanique statistique quantique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 513–523 (1941). [MF 6899]

This paper is based on two of De Donder's papers [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. 17, fasc. 5, 1–83 (1938); and the paper reviewed above]. The transition from classical statistical mechanics to quantum statistical mechanics is investigated, especially from the point of view of finding a connection between the usual function of distribution and that given by De Donder. *L. Infeld.*

Ufford, C. W. and Wigner, E. P. On the calculation of the distribution function. Phys. Rev. (2) 61, 524–527 (1942). [MF 6471]

In classical statistical mechanics it is important to know the density $g(x)$ of the probability that a given pair of the similar particles in the assembly be at a distance r apart. In principle $g(r)$ is derivable from the Boltzmann-Gibbs distribution, but at the cost of integrations involving all the other particles. Hence approximations are employed, as in the Debye-Hückel theory, in which certain expressions are replaced by averages and $g(x)$ obtained as the solution of an integral equation. The present paper considers the special case of the set of particles distributed on a circle of circumference L, the potential $v(x)$ between a pair at an arc distance x apart being given by $\beta v(x)=-\log\sin^2\pi x/L$. It happens that in this case both methods can be carried out explicitly and the Debye-Hückel approximation is thus checked against the theoretically exact derivation. The results are found to be in quite close agreement.

B. O. Koopman (New York, N. Y.).

Chang, T. S. and Ho, C. C. Arrangements with given numbers of neighbors. Proc. Roy. Soc. London. Ser. A. 180, 345–365 (1942). [MF 6933]

The problem of finding the number $g(N, n, X)$ of ways of arranging n atoms on N lattice points so that there are X pairs of nearest neighbors is attacked in the following way. The previous method [Chang, Proc. Cambridge Philos. Soc. 35, 265–292 (1939)] of finding this number for a linear chain is generalized to include any number of kinds of particles with correspondingly increased possibilities of selecting pairs of neighbors. The case of a lattice built up of points situated on parallel planes can be reduced to the previous one by identifying each different configuration in a plane as a different kind of "particle." The asymptotic evaluation of g for large arguments involves chiefly the solution of a determinantal equation, whose order increases rapidly with the number of points in the configurations on single planes. The case of two kinds of particles on two straight lines is worked out explicitly. The treatment of an infinite crystal would, however, involve a determinant of infinite order which cannot be resolved at present. *L. W. Nordheim.*

DECEMBER ISSUE IS AN INDEX WHICH HAS
BEEN PHOTOGRAPHED AT THE BEGINNING
OF THE VOLUME(S).